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Numerical Solution of One-Dimensional Heat Equation by Crank Nicolson Method

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ABSTRACT

In this paper we consider a Crank Nicolson algorithm for solving one-dimensional heat equation. In numerical analysis, the Crank–Nicolson method is a finite difference method used for numerically solving the heat equation and similar partial differential equations. The proposed method is quite efficient and is practically well suited for solving this problem. We compare numerical solution with the exact solution. The numerical solution is in good agreement with the exact solution. Finally, we investigate and compute the numerical results of proposed method for different step size. Several examples are given to verify the applicability and efficiency of the proposed method.

Keywords: Heat equation, Crank–Nicolson method, Numerical Solution Schemes, Application examples.

1. Introduction

The majority of practical design problems fall outside the reach of closed form solutions due to the complex and irregular form of structures, complexity of loading conditions, non-linearity and in homogeneity in properties of materials. For this reason, there is a growing interest in numerical methods for the solution of continuum mechanics problems. In such instances Finite difference Method (FDM) were used extensively in the 1960s. Now a day the FDM is a powerful tool for the approximate solution of differential equations governing diverse physical phenomena. Its use in industry and research is extensive, and indeed without it many problems in science and engineering would be incapable of solution. In finite difference, the mesh consists of rows and columns of orthogonal lines. In this paper we have used Crank Nicolson method to find numerical solution of heat equation. The Crank–Nicolson method is a finite difference method (FDM). It is used for finding numerical solution of engineering problems. Finite difference method is one of the numerical methods that are known as a highly applicable method in a lot of scientific fields. In many situations, findings an analytic solution to partial differential equations (PDE) is impossible, where numerical method are used to find approximate solutions. In real world, most of the problems in science and engineering are complicated enough that they can only be solved numerically. The heat equation plays a significant role in various physical problems.

Dehghan [1] developed numerical schemes for obtaining approximate solutions to the initial boundary-value problem for one-dimensional second-order linear parabolic partial differential equation with non-local boundary specifications replacing boundary conditions.

Mebrate [2] discussed Finite difference method and Finite element methods to compute the numerical solutions of a one dimensional heat equation together with initial condition and Dirichlet boundary conditions. Jamet [3] analyzed Stability and Convergence of a Generalized Crank-Nicolson Scheme on a Variable Mesh for the Heat Equation. Szyszka [4] presented an implicit finite difference method (FDM) for solving initial-boundary value problems (IBVP) for one-dimensional wave equation. He [5] applied the homotopy perturbation method to the search for traveling wave solutions of nonlinear wave equations. Abbasbandy [6] analyzed He's variational iteration method to the wave equations in an infinite one-dimensional medium and some non-linear diffusion equations. Noor and Mohyud-Din [7] applied Variational iteration method for solving higher-order nonlinear boundary value problems using He’s polynomials. Han *et al.* [8] proposed a finite-difference scheme for the one-dimensional time-dependent Schrödinger equation. In this paper, we have applied developed Crank Nicolson method to solve one dimensional heat equation.

This paper is organized as follows. In section 2 problem formulations, in section 3 numerical solution schemes, in section 4 Application examples, in section 5 discussion of results and in the last section the conclusion of the paper is presented.

2. Problem formulation

The one dimensional heat flow equation is defined by

$$\frac{\partial u(x,t)}{\partial t} = \beta \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1.1}$$

with initial and boundary conditions(IBC)

$$u(0,t) = T_0, u(l,.t) = T_l, u(x,0) = f(x) \tag{1.2}$$

where $u = u(x,t)$ is the dependent variable, T_0 is a constant temperature along time axis, T_l is a constant Temperature along $x = l$ and β is a constant coefficient. We have to solve equation (1.1) to replace

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the partial differential coefficients by the finite difference approximations: At the point u_j^i the finite difference approximation for $\dfrac{\partial^2 u(x,t)}{\partial x^2}$ is

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{h^2} \tag{1.3}$$

At the point u_{j+1}^i the finite difference approximation for $\dfrac{\partial^2 u(x,t)}{\partial x^2}$ is

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{u_{j+1}^{i+1} - 2u_{j+1}^i + u_{j+1}^{i-1}}{h^2} \tag{1.4}$$

Taking the average of (1.3) and (1.4) we have

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{u_{j+1}^{i+1} - 2u_{j+1}^i + u_{j+1}^{i-1} + u_j^{i+1} - 2u_j^i + u_j^{i-1}}{2h^2} \tag{1.5}$$

the forward difference approximation for $\dfrac{\partial u(x,t)}{\partial t}$ is

$$\frac{\partial u(x,t)}{\partial t} = \frac{u_{j+1}^i - u_j^i}{k} \tag{1.6}$$

Substituting (1.5) and (1.6) in (1.1) we get the following equation

$$\begin{aligned} &\delta(u_{j+1}^{i+1} + u_{j+1}^{i-1}) - (2\delta + 2)u_{j+1}^i \\ &= (2\delta - 2)u_j^i - \delta(u_j^{i+1} + u_j^{i-1}) \end{aligned} \tag{1.7}$$

Where $\delta = \dfrac{k}{h^2 \beta}$ and equation (1.7) is called the Crank-Nicolson finite difference scheme for heat flow equation.

3. Numerical Solution Schemes

Partial differential equations with specified initial and boundary conditions can be solved in a given region by replacing the partial derivatives by their finite difference approximations (FDA). The finite difference approximations to partial derivatives at a point (x_i, t_j) are described as : The xy -plane is divided into a network of rectangles of length $\Delta x = h$ and breadth $\Delta t = k$ by drawing the lines $x_i = ih$ and $t_j = jk$ parallel to x and y axes. The points of intersection of these lines are called grid points or mesh points or lattice points. The grid point (x_i, t_j) are denoted by u_j^i . The approximate value of u_j^i can be obtained by using equation (7) for all values of i and j .

4. Application Examples

In this section, we consider two linear heat flow problems to verify accuracy of the proposed Crank Nicolson Method. Numerical results are computed and the outcomes are represented by graphically.

Example 1: We consider one dimensional heat flow equation, $u_t = 2u_{xx}$ with initial and boundary conditions (IBC): $u(x,0) = 3\sin(\pi x) - 2\sin(5\pi x)$, $u(0,t) = 0, u(4,t) = 0$ on the interval $0 \le x \le 4$ and $0 \le t \le 4$. The exact solution of the given problem is $u(x,t) = 3e^{-2\pi^2 t}\sin(\pi x) - 2e^{-50\pi^2 t}\sin(5\pi x)$. The exact solution is obtained and shown in figure.1 (a) and approximate results are obtained and shown in figures 1(b) and 1(c) and 1(d).

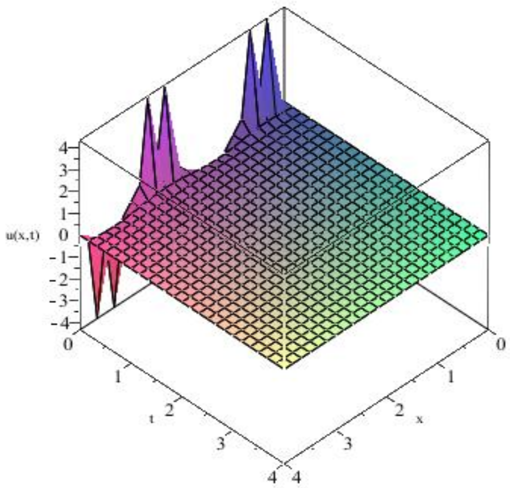

Fig1 (a): Exact solution $u(x,t)$ for different values of x and t of the given problem.

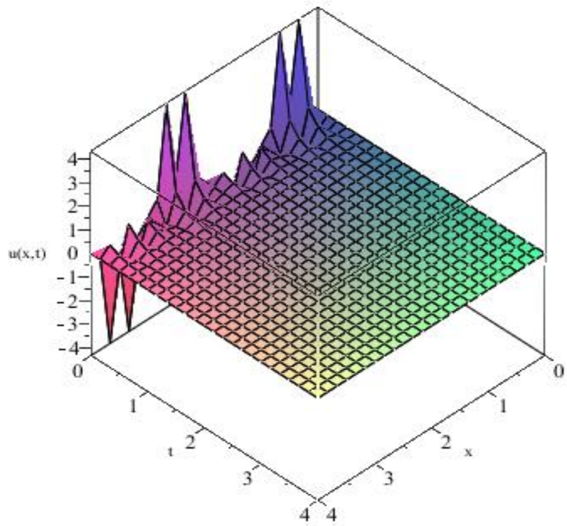

Fig1 (b): Approximate solution $u(x,t)$ for space step, h=0.04 and time step, k=0.04.

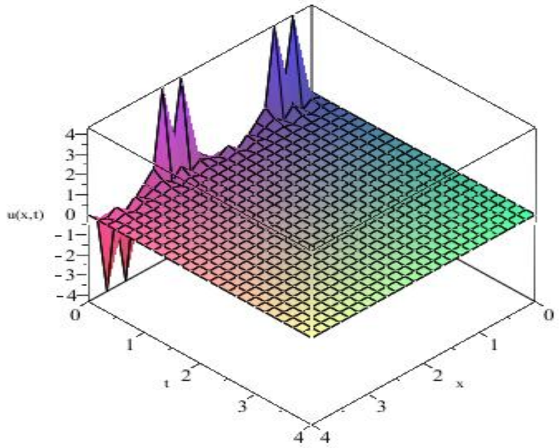

Fig1 (c): Approximate solution $u(x,t)$ for space step, h=0.03 and time step, k=0.03.

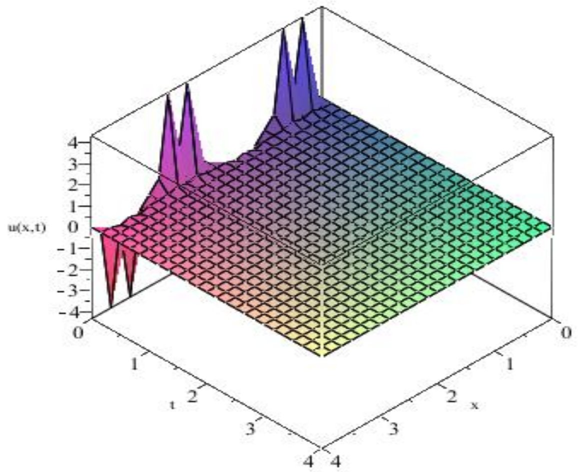

Fig1 (d): Approximate solution $u(x,t)$ for space step, h=0.025 and time step, k=0.025.

Example 2: We consider one dimensional heat flow equation, $u_t = 2u_{xx}$ with initial and boundary conditions (IBC): $u(x,0) = 10\sin(4\pi x), u(0,t) = 0, u(5,t) = 0$ on the interval $0 \le x \le 5$ and $0 \le t \le 5$. The exact solution of the given problem is given by $u(x,t) = 10e^{-32\pi^2 t}\sin(4\pi x)$ .The exact solution is obtained and shown in figure 2(a) and approximate results are obtained and shown in figures 2(b) and 2(c) and 2(d).

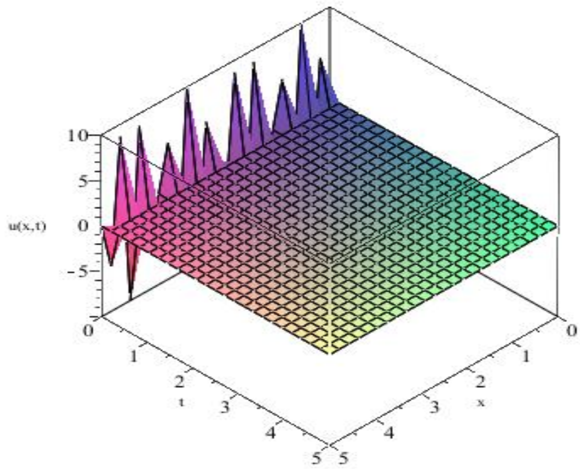

Fig2 (a): Exact solution $u(x,t)$ for different values of x and t of the given problem.

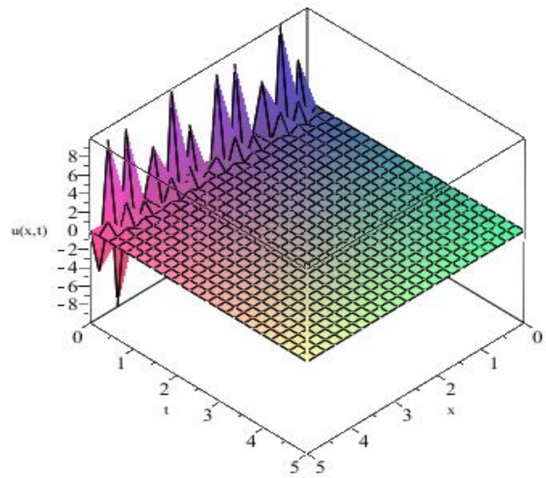

Fig2 (b): Approximate solution $u(x,t)$ for space step, h=0.04 and time step, k=0.04.

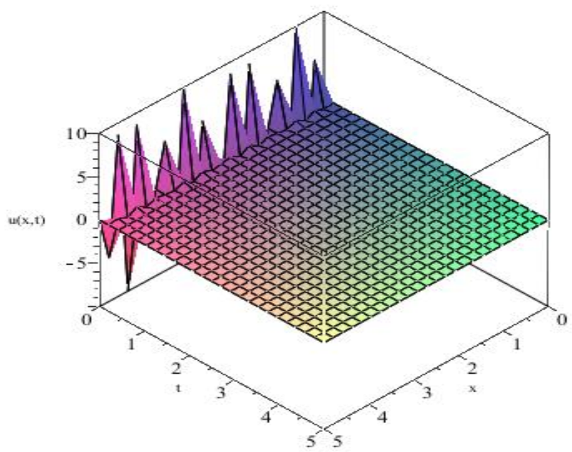

Fig2 (c): Approximate solution $u(x,t)$ for space step, h=0.03 and time step, k=0.03.

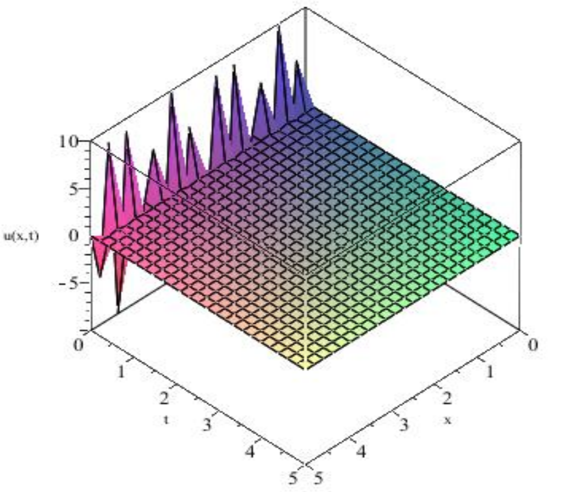

Fig2 (d): Approximate solution $u(x,t)$ for space step, h=0.025 and time step, k=0.025.

5. Discussion of results

The accuracy of the solution will depend on step size, h and k. In order to compare the approximate solution with the exact solution we consider two numerical examples. The effect of the space step h and time step k in the solution is shown in all figures: 1(a)-1(d) and figures: 2(a)-2(d). To obtain more accurate results, h should be small and k is necessarily very small. A numerical method is said to be convergent if $\lim\limits_{\substack{h\to 0\\ k\to 0}} \max\limits_{1\le n\le N} \left|U(x_n,t_n)-u(x_n,t_n)\right| = 0$.

Where $U(x_n,t_n)$ denotes the exact solution and $u(x_n,t_n)$ denotes the approximate solution. The approximated solution is evaluated by using Maple software for proposed numerical method at different step sizes. Finally we observe that the proposed Crank Nicolson method is converging faster if $\Delta x = h \to 0$ and $\Delta t = k \to 0$ and it is the most effective method for solving initial boundary value problems for partial differential equations (PDE).

6. Conclusion

In this paper, we have applied the Crank Nicolson algorithm to develop numerical solution scheme for the heat equation. We present two numerical tests to show the convergence and efficiency of the proposed numerical method. To find more accurate results to need the step sizes smaller for this proposed method. From the figures we can see the accuracy of the method for decreasing the step size h and k the graph of the approximate solution approaches to the graph of the exact solution. The numerical solutions obtained by the proposed method are in good agreement with exact solutions. From the study the proposed method was found to be generally more accurate and also the approximate solution converged faster to the exact solution. It may be concluded that the Crank Nicolson method is powerful and more efficient in finding numerical solutions of initial boundary value problems (IBVP) of heat equation.

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