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An Adaptive Mesh Refinement Scheme for the Solution of Mixed-Boundary Value Problems

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ABSTRACT

Some numerical simulations of multi-scale physical phenomena consume a significant amount of computational resources, since their domains are discretized on high resolution of meshes. An enormous wastage of these resources occurs in refinement of sections of the domain where computation of the solution does not require high resolutions. This problem is effectively addressed by adaptive mesh refinement (AMR), a technique of local refinement of a mesh only in sections where needed, thus allowing concentration of effort where it is required. Sections of the domain needing high resolution are generally determined by means of a criterion which may vary depending on the nature of the problem. Fairly straightforward criteria could include comparing the solution to a threshold or the gradient of a solution, that is, its local rate of change to a threshold. The objective of this paper is to develop an adaptive mesh refinement algorithm for finite difference scheme using potential function approach of fourth order bi-harmonic partial differential equation. In the AMR algorithm developed, a mesh of increasingly fine resolution permits high resolution computation in sub-domains of interest and low resolution in others. Then, the AMR scheme has been applied to solve a mixed boundary value elastic problem. In this work, the gradient of the solution has been considered as the criterion determining the regions of the domain needing refinement. Also the same problem is solved by classical Finite Difference Method (FDM) approach which uses uniform mesh over the whole numerical field. Finally, the solutions of both methods are presented as a comparative study to visualize the superiority of adaptive mesh refinement FDM technique over classical FDM technique. This analysis of superiority is done on basis comparison of solutions of both techniques with well known published results.

Keywords: Adaptive mesh refinement; Potential function approach, Classical finite difference method.

1. Introduction

Elasticity is now a classical subject and its problems are even more classical. But somehow these stress analysis problems are still suffering from a lot of shortcomings. Two factors may really be responsible for it. Both these factors involve management of the boundary of elastic problems: one is the boundary conditions and other is the boundary shape. There are various methods available for the solution of partial differential equations, which are needed for the stress analysis of structures. The FDM is one of the oldest numerical methods known for solving PDE's. The difference equations that are used to model governing equations in FDM are very simple to computer code and the global coefficient matrix that is produced by FDM possesses a banded structure, which is very effective for good solution. In spite of these characteristics, the necessity of the management of boundary shape has lead to the invention of the FEM and it's over whelming popularity, specifically because of the side by side development of high power computer machines. Of course, the adaptations of the FEM relieved us from our major inability of managing odd boundary shapes but we are constantly aware of its lack of sophistication and doubtful quality of the solutions so obtained. That is why FDM is chosen as solving method over FEM. There is present another factor of impediment to quality solutions of elastic problems is the treatment of the transition in boundary conditions. Several attempts were made to overcome both these two difficulties faced in the management of boundaries by FDM [1-2] and successfully overcome against these difficulties. But, using these [1-2] procedures, FDM simulations of some multi-scale physical phenomenon consumes a significant amount of computational efforts and resources because their domain are discretized on high resolution of meshes to achieve a good solution. To reduce the computational efforts and resources, several adaptive mesh refinement algorithms [3-14] have been developed over the last thirty years. But all such studies have limited application of solving the problems of either heat transfer or fluid mechanics. Moreover, the governing equation for the problems is either second order or lower. Mesh refinement is desirable to improve spatial solution. However, the uniform mesh refinement is not perfect for the applications of which the solution may need different resolutions for different regions. For example, for the mixed-boundary value problems, fine resolution is typically required for regions boundary. But for the evolving stress analysis of elastic fields with complicated structures, AMR techniques are more preferred to locally increase mesh densities in the

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regions of interest, thus saving the computer resources. The strategies of AMR can fall into two categories from the viewpoint of way of multi-resolution fulfilled. The first category includes these adaptive algorithms involved local mesh/stencil refinement. In these algorithms, either the existing mesh is split into several smaller cells or additional nodes are inserted locally, thus obtained the *h*-refinement. This group can be further categorized by the mesh type, i.e. hierarchical structured grid approach [3, 5-6] and unstructured mesh refinement approach [4]. The second category of adaptive algorithms involves global mesh redistribution. These methods move the mesh pint inside the domain in order to better capture the dynamic changes of solution and usually referred as moving mesh method or *r*-refinement [12-14] and at present, this category has application in FEM only. In this paper, AMR technique of fourth order bi-harmonic PDE is developed based on *h*-refinement by splitting the existing mesh into smaller cells. This approach is established on regular Cartesian meshes and at fine/coarse cell interfaces, special treatment is required for the communications between the meshes at different levels [15].

2. Governing Equations

Stress analysis in an elastic body is usually a three dimensional problem. But in most cases, the stress analysis of three-dimensional bodies can easily be treated as two-dimensional problem, because most of the practical problems are often found to conform to the states of plane stress or plane strain. In case of the absence of body forces, the equations governing the three stress components σ_x, σ_y and σ_{xy} under the states of plane stress or plane strain are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \tag{2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \tag{3}$$

Substitution of the stress components in Eq.(1-3) by their relations with the displacement components u and v make Eq.(3) redundant and transform Eq.(1) and (2) to

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{1-\mu}{2}\right)\frac{\partial^2 u}{\partial y^2} + \left(\frac{1+\mu}{2}\right)\frac{\partial^2 v}{\partial x.\partial y} = 0 \tag{4}$$

$$\frac{\partial^2 v}{\partial y^2} + \left(\frac{1-\mu}{2}\right)\frac{\partial^2 v}{\partial x^2} + \left(\frac{1+\mu}{2}\right)\frac{\partial^2 u}{\partial x.\partial y} = 0 \tag{5}$$

The problem thus reduces to finding u and v in a two dimensional field satisfying the two elliptic partial differential Eq.(4) and (5). Further the problem is reduced to the determination of a single function ψ instead of two functions u and v, simultaneously, satisfying the equilibrium Eq.(4) and (5) [9-10]. In this formulation, as in the case of Airy's stress function φ [7], a potential function $\psi(x,y)$ is defined in terms of displacement components as

$$u = \frac{\partial^2 \psi}{\partial x.\partial y} \tag{6}$$

$$v = -\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2 \psi}{\partial y^2} + \left(\frac{2}{1+\mu}\right)\frac{\partial^2 \psi}{\partial x^2}\right] \tag{7}$$

When the displacement components in the Eq.(4) and (5) are substituted by Eq.(6) and (7), Eq.(4) is automatically satisfied and the condition that ψ has to satisfy becomes

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2.\partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \tag{8}$$

Therefore, the problem is now formulated in such a fashion that a single function ψ has to be evaluated from bi-harmonic Eq.(8), satisfying the boundary conditions specified at the boundary.

2.1 General Boundary Condition

The boundary conditions at any point on an arbitrary shaped boundary are known in terms of the normal and tangential components of displacement, u_n and u_t and of stress σ_n and σ_t. These four components are expressed in terms of u, v, σ_x, σ_y, σ_{xy}, the components of displacement and stress with respect to the reference axes x and y of the body as follows:

$$u_n = u.l + v.m \tag{9}$$

$$u_t = v.l - u.m \tag{10}$$

$$\sigma_n = \sigma_x.l^2 + \sigma_y.m^2 + 2\sigma_{xy}.l.\,m \tag{11}$$

$$\sigma_t = \sigma_{xy}.(l^2 - m^2) + (\sigma_y - \sigma_x).l\,m \tag{12}$$

The boundary conditions at any point on the boundary are specified in terms of any two known values of u_n, u_t, σ_n and σ_t. In order to solve the mixed boundary-value problems of irregular-shaped bodies using present formulation, the boundary conditions are required to be expressed in terms of ψ. This can be done substituting the following expressions of the components of displacement and stress into Eq.(9) to (12).

$$u = \frac{\partial^2 \psi}{\partial x.\partial y} \tag{13}$$

$$v = -\left[\left(\frac{1-\mu}{1+\mu}\right)\frac{\partial^2 \psi}{\partial y^2} + \left(\frac{2}{1+\mu}\right)\frac{\partial^2 \psi}{\partial x^2}\right] \tag{14}$$

$$\sigma_x = \frac{E}{(1+\mu)^2}\left[\frac{\partial^3 \psi}{\partial x^2 \partial y} - \mu\frac{\partial^3 \psi}{\partial y^3}\right] \tag{15}$$

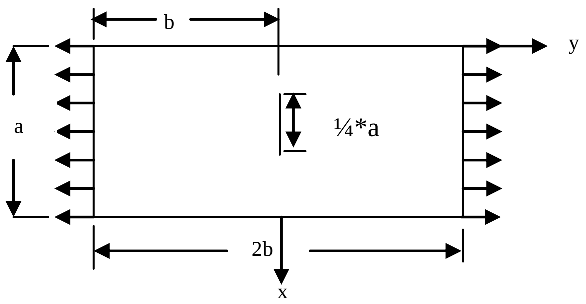
$$\sigma_y = -\frac{E}{(1+\mu)^2}\left[\frac{\partial^3 \psi}{\partial y^3} + (2+\mu)\frac{\partial^3 \psi}{\partial x^2 \partial y}\right] \tag{16}$$

$$\sigma_{xy} = \frac{E}{(1+\mu)^2}\left[\mu\frac{\partial^3\psi}{\partial x^2\partial y} - \frac{\partial^3\psi}{\partial x^3}\right] \qquad (17)$$

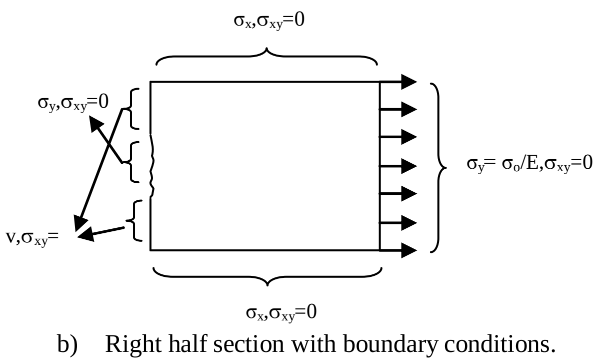
It is evident from the expressions of boundary conditions Eq.(9) to (12) that no matter what combinations of two conditions are specified on the boundary, the whole range of conditions that ψ has to satisfy Eq.(8) within the body and any two of the Eq.(9) to (12) at points on the boundary can be expressed as finite difference equations in terms of ψ(x,y).

2.2 Model Problem and its Boundary Conditions

A model problem chosen for this study is shown in Fig. 1. It is simple plate with an embedded crack at center of the plate. The length a/b=1, while crack length is one fourth of ‘a’ or ‘b’. The boundary conditions are expressed in terms of stresses and displacement in normal and tangential directions. The problem treated here is, therefore, obviously a two dimensional problem with mixed boundary conditions. Due to the material and loading symmetry only right half section is taken for analysis. The top and bottom edge is free and thus obvious σ_x, σ_{xy}=0. At the right edge there is present a normal tensile stress thus σ_y =P=2e^{-4} and σ_{xy}=0. And at the left edge, σ_{xy}=0 and v=0 except crack position where σ_y =0 and σ_{xy}=0. Here, the stress components are normalized by the young modulus, E (e.g. $\sigma_y=\sigma_o/E$). For this problem the Poisson’s ratio is taken as μ=0.3.



a) Physical problem.

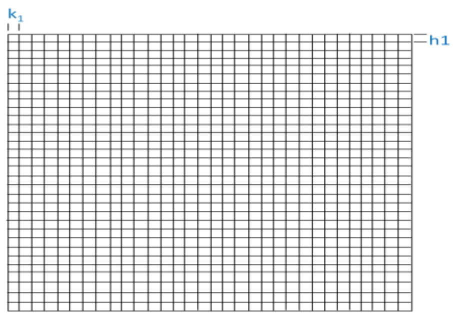


b) Right half section with boundary conditions.

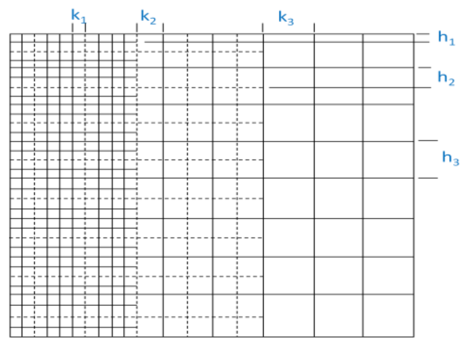
Fig.1 Physical geometry of the elastic problem and its boundary conditions.

3. Solution of the Problem

For the solution of the problem, a two dimensional mesh is generated based on rectangular coordinate system. The function ψ from the governing Eq.(8) is evaluated at various mesh points inside the body using central difference formula. The function ψ from the boundary conditions is evaluated in the same manner by forward and backward difference formula at the boundary points depending on the physical boundary. A FORTRAN code has been developed to investigate various aspect of the problem. The full procedure of the management of boundary conditions has already been discussed in the papers [17-18]. But that procedure gives better result when the meshes are discretized base on uniform grid throughout the domain. Under AMR technique, for this particular problem the domain is discretized as follows [fig.2]. Uder classical FDM there is only one sizes mesh of length h_1 and k_1 in direction x- and y- respectively, whereas, the AMR technique has three different size of mesh. The high resolution of meshes under AMR technique is taken in the vicinity of the crack. Here, in this thesis the discretization is done by such a way that both methods have almost equal no. of nodal points. To satisfy governing equation and boundary condition over the whole field, some new stencils of governing equation and boundary conditions have been developed. The details of applications procedure of these stencils is given in [15].



a) Discretization under Classical FDM.



b) Discretization under AMR technique.

Fig.2 Discertization of the domain under classical FDM and AMR technique.

4. Results and Discussions

Following the procedure stated above and taking mesh size 0.0167 unit for classical FDM and for AMR that is taken as 0.0083 for smallest mesh, 0.0167 for medium mesh and 0.033 for largest mesh [fig.2], results are obtained by both classical FDM and AMR technique of half section of the problem. In fig.3 displacement in x-direction is shown for two section namely y/b=0.0 and 1.00, and it shows that displacement obtained by both methods is same. But at the tip of the crack, the AMR technique gives better results than classical FDM due to high resolution. In fig.3 displacement in y-direction is shown for two sections and shows that results are almost same, although, AMR shows a smaller value than classical FDM. But the deviation is not very much significant. From this study, it can be concluded that the AMR technique has no improvement of results over classical FDM in terms of displacement of the model.

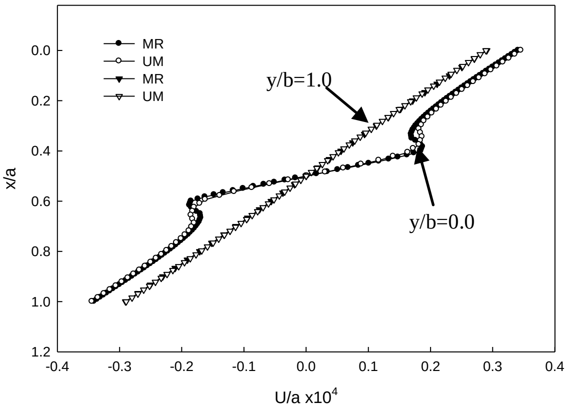


Fig.3 Comparison of the results for normalized displacement (U/a) obtained by mesh refinement (MR) technique and uniform mesh (UM) technique with almost equal no. of nodal points.

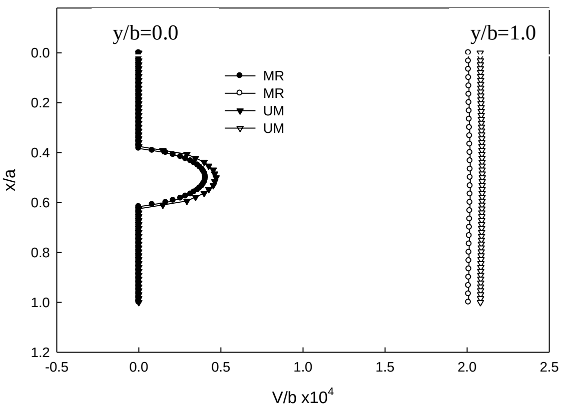


Fig.4 Comparison of the results for normalized displacement (V/b) obtained by MR technique and UM technique with almost same no. of nodal points.

The most significant component of stresses of this problem is stress in y-direction i.e. σy, which is shown in fig.5. From this graph it is seen that for almost equal no. of nodal points the AMR technique gives a higher value of σ_y than classical FDM. A validation of these results can be shown as follows. Analytical solution of this considered problem is given in literature [19]. Consider a mode I crack of length 2*a* in the infinite plate of Fig.6. By using complex stress functions, it has been shown that the stress field on a *dx dy* element in the vicinity of the crack tip is given by

$$\sigma_y = \sigma\sqrt{\frac{a}{2r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{18}$$

This equation tells that at the tip of the crack i.e. at r=0 and θ=0 the value of stress should be infinite. But for our case, it is seen that [fig.5] the stress is finite value and in classical method, it is only 2.25 times of applied stress (σ_y/E= 2e^{-4}) and in AMR technique, it is 3.5 times of the applied stress. An explanation of this phenomenon can be given as, in FDM method; it is never possible to take a node at the tip of the crack, because one always stays behind the tip of the crack by a distance of half mesh length. By taking r=0.5* mesh length and θ=0 from eq.18, it is found that the stress value should be 3.74 times of the applied stress and this is very close to the results of AMR technique. So it is verified that AMR technique is better than classical FDM for the solution of problems which required different level of resolution at different region of the problem. The comparison of results for σ_y at other section of the plate is shown in fig.7 and it is seen that, at other section results is almost same in both methods. The comparison of stress component in x-direction is shown in fig.8 and it is seen that in AMR technique results is higher than that of classical FDM.

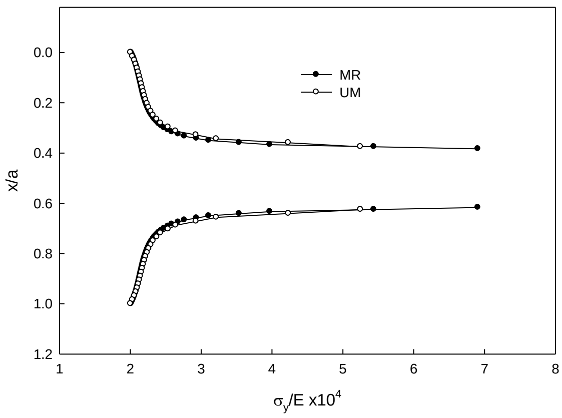


Fig.5 Comparison of the results for normalized normal stress (σ_y/E) obtained by MR technique and UM technique with almost equal no. of nodal points.

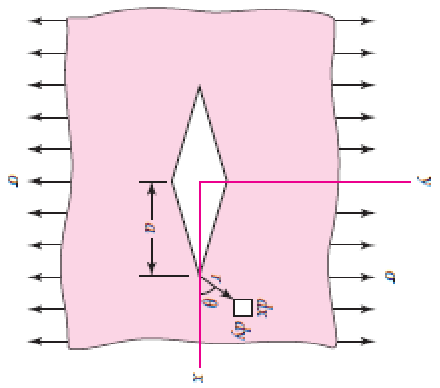


Fig.6 Plate with embedded crack under uniform tension.

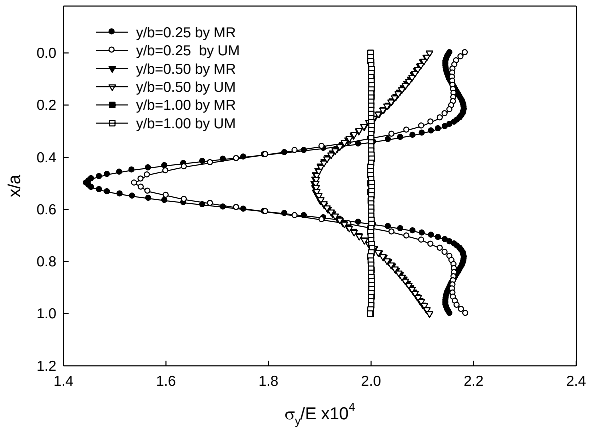


Fig.7 Comparison of the results for normalized normal stress (σ_y/E) obtained by MR technique and UM technique at different section of plate.

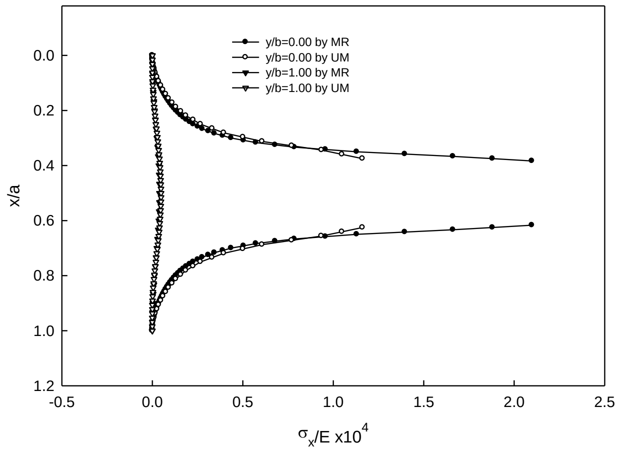


Fig.8 Comparison of the results for normalized normal stress (σ_x/E) obtained by MR technique and UM technique with almost equal no. of nodal points.

5. Conclusions

The adaptive mesh refinement technique, for the problem which need different resolution at different region of the domain, described here in this paper is already proven to be advantageous over conventional finer mesh generation technique of the finite difference method. Redistribution of nodes in AMR technique improves the accuracy of the solutions near stress concentration zone. The AMR technique requires a lesser amount of computational memory due to lesser number of unknown parameters. Thus the AMR technique could be a strongest weapon to solve problem which cannot be solved by uniform mesh generation technique due to memory shortage of computational resources.

NOMENCLATURE

- E : Modulus of Elasticity, GPa
- μ : Poisson ratio
- ψ : Displacement potential function
- σ_x : Normal stress component along x-direction
- σ_y : Normal stress component along y-direction
- σ_{xy} : Shear stress component in the xy plane
- σ_n : Stress component normal to boundary
- σ_t : Stress component tangential to boundary
- u : Displacement component along x-direction
- v : Displacement component along y-direction
- l, m : Direction cosine of the normal at any physical boundary point
- u_n : Displacement component normal to boundary
- u_t : Displacement component tangential to boundary

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