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Analysis of Stress Distribution between Interacting Planar Cracks

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ABSTRACT
A hand-made numerical program based on Body Force Method (BFM) applicable to planar crack problems is developed. Numerical analysis is carried out for arbitrary shaped coplanar 3D cracks interacting each other in an infinite solid. As a fundamental solution, a stress field at an observation point induced by a body force doublet applied at a source point in an infinite elastic medium is employed. In the present analysis, a planar triangular element is used to cover the total crack surface. Special crack tip elements considering the stress singularity at a crack front are employed to the crack front element. The crack problem is formulated as hypersingular boundary integral equations with unknown distribution of the body force doublet. Finally, the unknown distribution of the body force doublet is solved by transforming the boundary integral equation into a set of simultaneous equations. In addition to the theoretical background of the present method, several numerical results are shown graphically.

Keywords: Body force method, Mixed-mode SIF, Crack interaction, Triangular element.

1. Introduction

A crack to crack interaction exhibits a major influence on behavior of crack growth. Multiple cracking is one of the most common problems in engineering structures and may caused catastrophic or structural failures in aircraft, ship, bridge, automotive components and machine parts. The interactions of crack change the SIF and stress distribution near the crack fronts. The SIF and stress distribution because of crack interaction not only depended on the number of cracks but also the size, shape, distance between cracks. The analysis of SIF and stress distribution due to crack interaction is very important and useful in evaluating the strength and safe life prediction of engineering materials structures. Several numerical methods have been applied for the SIF and stress distribution analysis of multiple cracks, such as finite element method [1], enriched meshless method [2], integral equation method [3], boundary element method [4], boundary collocation method [5], Lagrangian finite difference method [6], alternating iteration method [7] and body force method. Among these methods FEM and BEM are much more general than other methods. But the main problem in BEM and FEM is that, it is indispensable to divide a whole domain or surface into several segments. Furthermore, it is not convenient to simulate the crack propagation due to need for re-meshing near crack tips. In this paper body force method (BFM) has been used for the analyzing multiple cracks. Nisitani proposed BFM as a method of numerical stress analysis [8]. Compared with the FEM, the BFM has some advantages in solving elastic fracture problems. The reason is due to the fact that the BFM only contains the boundary discretization of the problem domain and a more accurate result could be obtained with a lesser effort. Nisitani et al. [9] first applied the body force method to investigate problems of an elliptical crack or a semi elliptical crack in an infinite plate under tension. After that Murakami et al. [10] applied BFM for the analysis of interaction between semi-elliptical surface cracks in an semi-infinite elastic body under tension and bending. Isida et al. [11] applied BFM for the first time to analyze two parallel elliptical cracks under mixed-mode stress state at the infinity. Isida et al. [12] again applied BFM for the analysis of multiple semi-elliptical surface cracks in semi-infinite solid under tension. Wang et al. [13] discussed numerical solutions using singular integral equation of the BFM for 3D rectangular crack problems. Later Noda et al. [14] applied singular integral equation method based on BFM for the calculation of SIF of interacting semi-elliptical surface cracks. In BFM, each research has been carried out by developing the special numerical program suitable for each problem. Therefore, a versatile stress analysis program applicable to arbitrary shaped 3D interacting cracks under mixed-mode loading has not been developed until today.

In this research a versatile numerical program has been developed for the analysis of mixed-mode planar crack. The 3D crack problem is formulated in terms of singular integral equations with singularity of the order of r^{-3}, where r is the distance between source and reference points. The stress field induced by a body force doublet in an infinite body is used as the fundamental solution. The unknown functions are approximated by the product of fundamental density function and weight function. The approximate solution is obtained easily by providing the triangular mesh data and boundary conditions. By using the developed program, the coplanar interactive surface cracks in an infinite solid

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subjected to remote tension and bending loads are computed.

2. Fundamental principle of BFM

The body force method (BFM) is directly based on this principle of superposition. The essence of BFM is to transform a given elastic problem to an equivalent problem of an infinite domain in which body forces are embedded. BFM is classified as an indirect BEM since the stress and displacement at a reference point is expressed in terms of densities of body force. The fundamental concept of BFM is to express the elastic fields of a problem by means of the principle of superposition. A solution of an elastic problem should satisfy three conditions of i) equilibrium condition, ii) compatibility condition and iii) boundary condition. Since the elastic fields are superposable, in BFM the elastic field due to a point force under investigation can be expressed by superposing some specific known elastic fields. In the BFM the specific known elastic fields, which can be written in a closed form, are superposed with some unknown weight-magnitudes as follows

$$\boxed{\begin{array}{c}\text{Elastic field}\\ \text{under}\\ \text{investigation}\end{array}} = \sum \boxed{\begin{array}{c}\text{Specific}\\ \text{known solution}\\ \text{of elastic field}\end{array}} \times \boxed{\begin{array}{c}\text{Unknown}\\ \text{weight-}\\ \text{magnitudes}\end{array}} \quad (1)$$

If a given boundary condition is satisfied by adjustment of the unknowns in Eq. (1), the superposed elastic field is the solution of the given problem, because the solution of elastic problem is unique. Finally Eq. (1) is transformed into summation of simultaneous equations and unknown weight magnitudes to be determined through boundary conditions. If the boundary condition is satisfied exactly, it means that an exact solution is derived by BFM.

Consider an infinite body having a penny-shaped crack on *x*-*y* plane which is subjected to uniform tension at infinity. According to the body force method this problem is transferred as a combination of two problems. One is an infinite solid with uniform stress distribution at the infinity and another is an imaginary crack in an infinite solid along which the unknown force doublets is acting continuously as shown in Fig. 1.

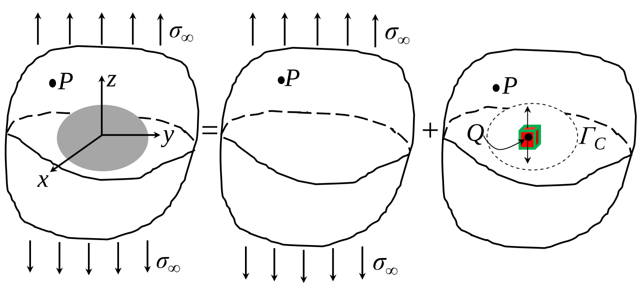

Fig.1 Uniaxial tension of an infinite solid with a single penny-shaped crack.

The stress component at an arbitrary point in an infinite solid with crack can be expressed as follows,

$$\sigma_{ij}(P) = \sigma_{ij}^{\infty}(P) + \iint_{\Omega_c} \sigma_{ij}^{kl}(P,Q)\rho_{kl}(Q)d\Omega_c(Q) \qquad (2)$$

Where $P(x,y,z)$ is a reference point, $Q(\xi,\eta,\zeta)$ is a source point, Ω_c is an imaginary crack surface, $\sigma_{ij}^{\infty}(P)$ represents the initial stress at the reference point, ρ_{kl} is the density of force doublet, $\sigma_{ij}^{kl}(P,Q)$ is the stress component at a reference point which is obtained by differentiation with respect to co-ordinate variable $l\ (l = \xi,\eta,\zeta)$ of the stress component σ_{ij} due to a unit magnitude of point force acting in the $k\ (k = x,y,z)$ direction at the source point [15]. In order to determine $\rho_{kl}(Q)$, it is required to approach P from the exterior of Γ to the surface of the crack. At the surface of the crack, the traction-free condition is applied. Thus Eq. (2) becomes the boundary integral equation for the determination of unknown density $\rho_{kl}(Q)$ in a complete infinite domain. Once the unknown densityfunction $\rho_{kl}(Q)$ is obtained, the stress at an arbitrary point P can be obtained by putting the $\rho_{kl}(Q)$ in Eq. (2). In such a way, an arbitrary boundary value problem is transformed into a form of integral equations with unknown density through the principle of superposition of the known fundamental solution.

3. Theoretical Analysis

In BFM, solution of any elasticity problem is transformed into a problem of a complete infinite domain without any crack. That is, a boundary of given problem is replaced by an equivalent imaginary boundary along which body force or body force doublets are embedded. In BFM, an elastic boundary value problem is transformed into the form of a boundary integral equation. Consider an infinite body having a penny-shaped crack as shown in Fig. 2 which is subjected to mixed mode loading at infinity.

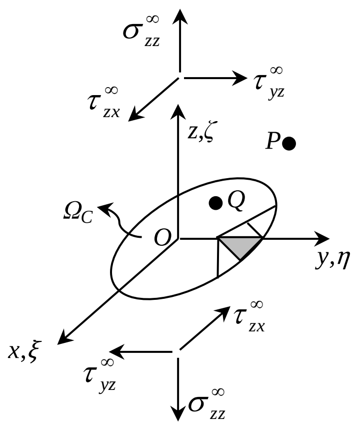

Fig.2 Crack in an infinite solid subjected to uniform stress at infinity.

Let σ_{zz}, τ_{zx} and τ_{yz} be the stress component due to the fundamental force doublets distributed over the crack surface and stress at the infinity. On the idea of the body force method, the problem is reduced to sets of integral equations in which the density of force doublets are unknowns to be determined. The stress components at any arbitrary point are as follows:

$$\sigma_{zz}(P) = \sigma_{zz}^{\infty}(P) + \iint_{\Omega_c} [\sigma_{zz}^{zz}(P,Q)\gamma_{zz}(Q) + \sigma_{zz}^{zx}(P,Q)\gamma_{zx}(Q) + \sigma_{zz}^{yz}(P,Q)\gamma_{yz}(Q)]d\Omega_c(Q) \quad (3)$$

$$\tau_{zx}(P) = \tau_{zx}^{\infty}(P) + \iint_{\Omega_c} [\tau_{zx}^{zz}(P,Q)\gamma_{zz}(Q) + \tau_{zx}^{zx}(P,Q)\gamma_{zx}(Q) + \tau_{zx}^{yz}(P,Q)\gamma_{yz}(Q)]d\Omega_c(Q) \quad (4)$$

$$\tau_{yz}(P) = \tau_{yz}^{\infty}(P) + \iint_{\Omega_c} [\tau_{yz}^{zz}(P,Q)\gamma_{zz}(Q) + \tau_{yz}^{zx}(P,Q)\gamma_{zx}(Q) + \tau_{yz}^{yz}(P,Q)\gamma_{yz}(Q)]d\Omega_c(Q) \quad (5)$$

Where $P(x,y,z)$ is a reference point, $Q(\xi,\eta,\zeta)$ is a source point, Ω_c is an imaginary crack surface, γ_{zz}, γ_{zx} and γ_{yz} are the density of standard force doublets.
In above equations the fundamental solutions are calculated from the following expressions.

$$\sigma_{ij}^{zz}(P,Q) = \left.\frac{\partial \sigma_{ij}^{Z}}{\partial \zeta}\right|_{Z=1} + A\left\{\left.\frac{\partial \sigma_{ij}^{X}}{\partial \xi}\right|_{X=1} + \left.\frac{\partial \sigma_{ij}^{Y}}{\partial \eta}\right|_{Y=1}\right\} \quad (6)$$

$$\sigma_{ij}^{zx}(P,Q) = \left.\frac{\partial \sigma_{ij}^{X}}{\partial \zeta}\right|_{X=1} + \left.\frac{\partial \sigma_{ij}^{Z}}{\partial \xi}\right|_{Z=1} \quad (7)$$

$$\sigma_{ij}^{yz}(P,Q) = \left.\frac{\partial \sigma_{ij}^{Y}}{\partial \zeta}\right|_{Y=1} + \left.\frac{\partial \sigma_{ij}^{Z}}{\partial \eta}\right|_{Z=1} \quad (8)$$

$$i,j = x,y,z; \; A = \frac{\nu}{1-\nu}$$

Where, $\sigma_{zz}^{kl}, \sigma_{zx}^{kl}$ and σ_{yz}^{kl} be the stress components due to the fundamental force doublets distributed over the crack surface. $\sigma_{ij}^{X}, \sigma_{ij}^{Y}$ and σ_{ij}^{Z} are the known expressions of stress components at point $P(x,y,z)$ due to concentrated forces acting at point $Q(\xi,\eta,\zeta)$ called Kelvin solution. The concrete from of fundamental solution are as follows:

$$\sigma_{zz}^{zz}(P,Q) = \frac{1-2\nu}{8\pi(1-\nu)^2}\left[\frac{1}{r^3} + 6\frac{{r_z}^2}{r^5} - 15\frac{{r_z}^4}{r^7}\right] \quad (9)$$

$$\sigma_{zz}^{zx}(P,Q) = \frac{3}{4\pi(1-\nu)} r_x r_z \left[\frac{1}{r^5} - 5\frac{{r_z}^2}{r^7}\right] \quad (10)$$

$$\sigma_{zz}^{yz}(P,Q) = \frac{3}{4\pi(1-\nu)} r_y r_z \left[\frac{1}{r^5} - 5\frac{{r_z}^2}{r^7}\right] \quad (11)$$

$$\tau_{zx}^{zz}(P,Q) = \frac{3(1-2\nu)}{8\pi(1-\nu)^2} r_x r_z \left[\frac{1}{r^5} - 5\frac{{r_z}^2}{r^7}\right] \quad (12)$$

$$\tau_{zx}^{zx}(P,Q) = \frac{1}{4\pi(1-\nu)}\left[\frac{E}{r^3} - \frac{3\nu {r_y}^2}{r^5} - \frac{15{r_x}^2{r_z}^2}{r^7}\right] \quad (13)$$

$$\tau_{zx}^{yz}(P,Q) = \frac{3}{4\pi(1-\nu)} r_x r_y \left[\frac{\nu}{r^5} - 5\frac{{r_z}^2}{r^7}\right] \quad (14)$$

$$\tau_{yz}^{zz}(P,Q) = \frac{3(1-2\nu)}{8\pi(1-\nu)^2} r_y r_z \left[\frac{1}{r^5} - 5\frac{{r_z}^2}{r^7}\right] \quad (15)$$

$$\tau_{yz}^{zx}(P,Q) = \frac{3}{4\pi(1-\nu)} r_x r_y \left[\frac{\nu}{r^5} - 5\frac{{r_z}^2}{r^7}\right] \quad (16)$$

$$\tau_{yz}^{yz}(P,Q) = \frac{1}{4\pi(1-\nu)}\left[\frac{E}{r^3} - \frac{3\nu {r_x}^2}{r^5} - \frac{15{r_y}^2{r_z}^2}{r^7}\right] \quad (17)$$

Where $r_x = x-\xi, r_y = y-\eta, r_z = z-\zeta, E = 1+\nu$
$r^2 = {r_x}^2 + {r_y}^2 + {r_z}^2$ and ν is the poisson's ratio.

In this analysis, the surface of the crack is expressed by the aggregation of planar triangles and in each triangle the density of the force doublets is assumed at constant. The triangles which are placed at the crack front, the basic density function is considered as shown in Fig. 3. Density of standard force doublets γ_{zz}, γ_{zx} and γ_{yz} are expressed by the product of basic density function $\sqrt{h(\xi,\eta)}$ and weight functions $W_{zz}(\xi,\eta), W_{zx}(\xi,\eta)$ and $W_{yz}(\xi,\eta)$ respectively.

$$\gamma_{zz}(Q) = \sqrt{h(\xi,\eta)}W_{zz}(\xi,\eta) \quad (18)$$

$$\gamma_{zx}(Q) = \sqrt{h(\xi,\eta)}W_{zx}(\xi,\eta) \quad (19)$$

$$\gamma_{yz}(Q) = \sqrt{h(\xi,\eta)}W_{yz}(\xi,\eta) \quad (20)$$

$$h(\xi,\eta) = \frac{|x_2y_1 - y_2x_1 + y_{21}\xi - x_{21}\eta|}{\sqrt{{y_{21}}^2 + {x_{21}}^2}} \quad (21)$$

Where $y_{21} = y_2 - y_1$ and $x_{21} = x_2 - x_1$

When the crack is free of traction, σ_{zz}, τ_{zx} and τ_{yz} are zero at the same time, when $P \to P^{\Omega c}$.

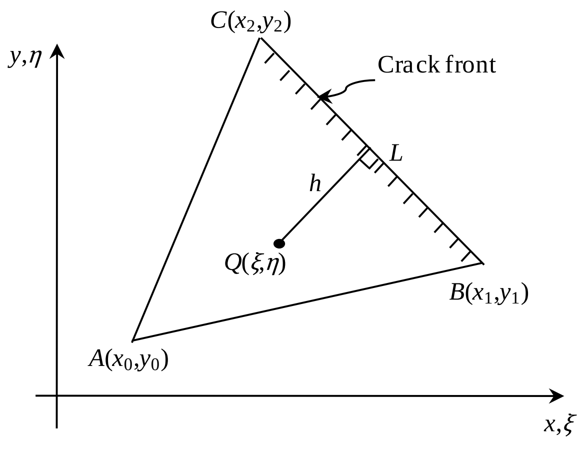

Fig. 3 Planar triangle surface element with force doublets (shaded triangle in Fig. 2).

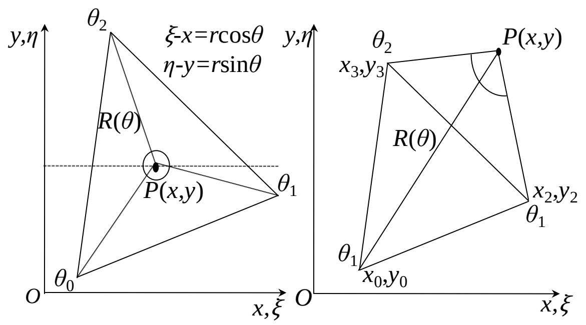

Fig. 4 Parameter transformation from cartesian to polar coordinates.

After solving the simultaneous equations expressing the boundary condition, the value of weight functions are obtained. Weight function $W_{zz}(\xi,\eta)$ is responsible for the mode-I SIF K_{I} and $W_{zx}(\xi,\eta), W_{yz}(\xi,\eta)$ are responsible for the K_{II} and K_{III} at crack front. Hyper singularity occurs, when observation point is on crack surface i.e. $x=\xi$ and $y=\eta$. In order to remove this singularity polar transformation has been used. For the calculation of integral equations numerically, integration limits varies depending on the position of observation points. We can calculate $\theta_0, \theta_1, \theta_2$ from the vertices of triangle and observation point. For θ_3 if the observation point is inside of triangle then $\theta_3 = \theta_0 + 2\pi$, if outside of triangle then $\theta_3 = \theta_0$ as shown in Fig. 4. Here, $R(\theta)$ means a distance between reference point $P(x,y)$ and a point on the prospective boundary of the triangle as shown in Fig. 4.

4. Numerical results and discussion

In order to employ the BFM to deduce the interaction effects of multiple cracks on their SIF values, it is necessary to correlate the SIF values of single crack predicted by BFM with published data. Initially, a single crack subjected to mixed-mode loading at infinity is considered to illustrate the effectiveness of the present analysis. The normalized SIF is compared with the literature solution. In this method desirable results can be obtained with relatively coarse pattern as the SIFs values are obtained by extrapolating the obtained results for different number of triangles NT. The accuracy of the SIF calculation was satisfactorily examined for rectangular, penny-shaped and elliptical planar cracks embedded in an infinite elastic body. Later the numerical results of stress distribution for the interference effects between two coplanar penny-shaped cracks, elliptical cracks, rectangular cracks, penny-shaped and elliptical cracks, penny-shaped and rectangular cracks, and elliptical and rectangular cracks has been carried out. The obtained numerical results shows that the numerical approach presented in the present study is simple, yet very user friendly for analyzing the interference effect of arbitrary shaped multiple cracks in plane elasticity. In order to verify the numerical accuracy, SIF calculation was examined for elliptical cracks with different aspect ratio embedded in an infinite elastic body. In this paper, for all cases the poisson's ratio was set at $\nu = 0.3$. In demonstrating the results of stress intensity factor SIFs, the following dimensionless F_{n} will be used.

$$F_{\mathrm{n},b} = \frac{(K_{\mathrm{n}})_b}{\sigma_{ij}^{\infty}\sqrt{\pi b}} \tag{22}$$

Where, $i,j = x,y,z$ and n=I,II,III.

The normalized SIFs of elliptical crack is calculated for different aspect ratio and plotted in Fig. 5. The normalized mode-I stress intensity factor F_{I} is compared with the exact solutions. It is found that the present solutions are almost coincides with the exact solution. In present research the surface of the corresponding crack was divided with regularly distributed number of triangles NT. In this method desirable results can be obtained with relatively coarse pattern as the SIFs values are obtained by extrapolating the obtained results for different number of triangles.

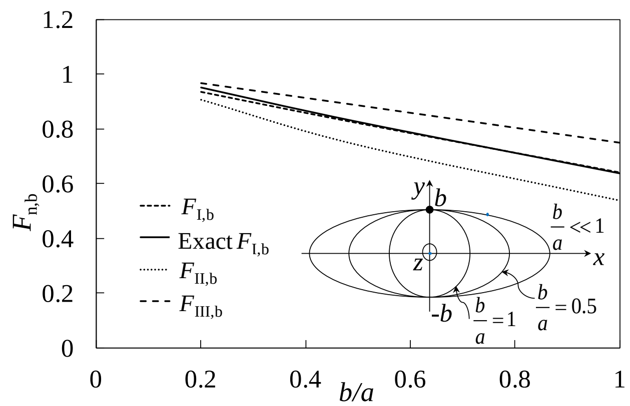

Fig. 5 Normalized mixed-mode SIF of a elliptical crack.

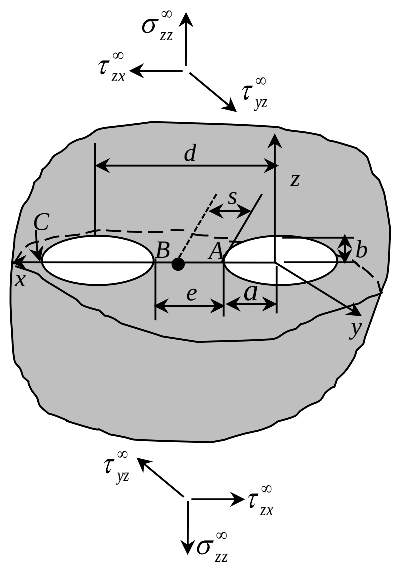

Fig. 6 Interference of two cracks under mixed-mode loading at infinity.

Fig. 6 shows the two coplanar cracks in an infinite elastic medium under remote loading at infinity along *zz*, *zx* and *yz* direction. The distance *e* between A and B can be varied. Both cracks surface is meshed in a same number of triangles NT. The ratio of minor and major axes of elliptical cracks are varied. When the radius of each crack is *a*=*b*=1.0, the normalized τ_{zx} distribution along *x*-axis for different values of *e* is shown in Fig. 7. It is found that the smaller the distance *e* between A and B, the greater the change of stress distribution along line AB. Near point C, the influence of the distance between A and B on the stress intensity factor is smaller. When *e/a*=0.5 and for the same radii of penny-shaped crack, the stress distribution is higher than stress distribution for other values of *e/a*. But as the ratio of *d/a* is increases the stress distribution is decreases. When the ratio of *e/a*=2.0, the stress distribution is same manner like the stress distribution of single penny-shaped crack as shown in Fig. 7. The interference analysis between elliptical cracks under mixed stress state at the infinity is also carried out. The normalized τ_{zx} distribution along *x*-axis between elliptical cracks for different values of *e* is shown in Fig. 8. The normalized stress distribution has the same trend like the normalized stress distribution between penny-shaped cracks.

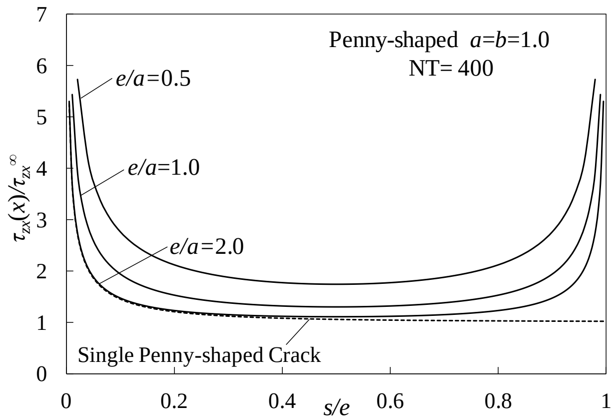

Fig. 7 Interference analysis between penny-shaped cracks.

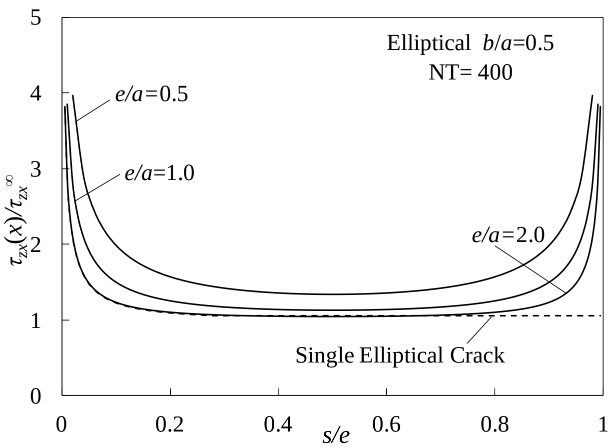

Fig. 8 Interference analysis between elliptical cracks.

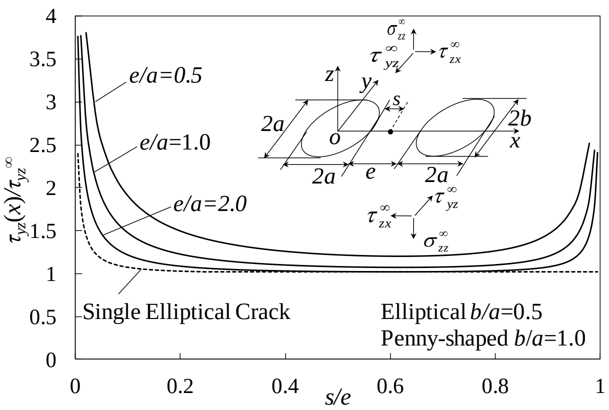

Fig. 9 Interference analysis between penny-shaped and elliptical cracks.

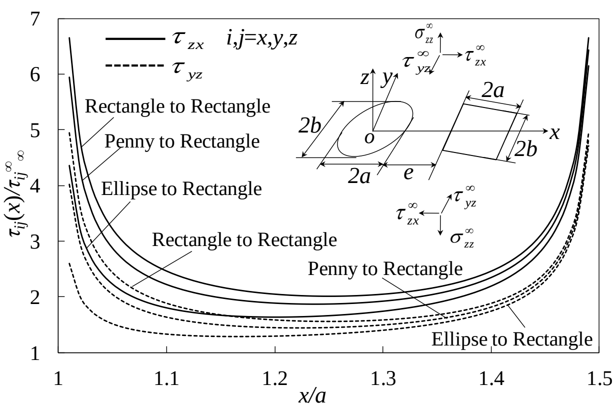

Fig. 10 τ_{ij} distribution between various shaped planar cracks (*e/a*=0.5).

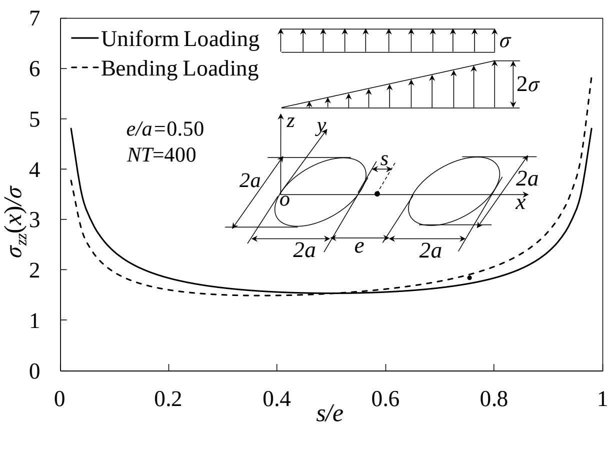

Fig. 11 Distribution of σ_{zz} along *x*-axis for uniform stress and bending loads.

Fig. 9 shows the interference effect between penny-shaped and elliptical cracks for different distance between center of cracks and fixed number of NT. Here aspect ratio *b/a* of the cracks are taken as 0.5. But any elliptical cracks with different aspect ratio can be analyzed. The crack tip difference *e* is varied from 0.5 to 2.0. For different values of *e/a*, as the distance between centers of cracks increases the magnitude of stress distribution decreases. Fig. 10 shows the interference effect between different shaped cracks with fixed distance of *e/a*. The number of triangular elements for each crack was fixed at 400. Among compared for both cases, the rectangle to rectangle crack combination showed the largest interference effect. The interference analysis of cracks under bending load is also analyzed. Fig. 11 shows the interference between penny-shaped cracks under uniform stress and bending loads respectively for fixed distance between center of cracks and fixed NT.

5. Conclusion

The SIFs of cracks were evaluated using developed numerical program based on BFM. In this study, the effect of interaction between different shaped cracks on the SIFs and stress distribution was investigated. The problem was formulated as an integral equation on the idea of BFM. The unknown functions were approximated by the product of fundamental density function and weight function. The approximate solution is obtained easily by providing the triangular mesh data and boundary conditions. The interaction is influenced not only the relative position but also by the relative shape and length of crack. The interaction between two cracks becomes large as the distance between the cracks become small. If the crack length difference is greater than a certain level, there is no interaction effect on the stress distribution. From the obtained results it was found that the developed numerical program is applicable for analyzing interacting between arbitrary shaped planar cracks under any loading condition. Any kinds and any number of 3D planar cracks under any loading condition can be solved effectively only by providing input data.

NOMENCLATURE

$P\,(x,y,z)$	: Reference point
$Q(\xi,\eta,\zeta)$	: Source point
γ_{ij}	: Density of standard force doublets (*i*,*j*=*x*,*y*,*z*)
$\sqrt{h(\xi,\eta)}$	: Basic density function
$W_{ij}(\xi,\eta)$	: Weight functions (*i*,*j*=*x*,*y*,*z*)
$R(\theta)$	: Distance between reference point and a point on the prospective boundary
σ_{ij},τ_{ij}	: Stress components (*i*,*j*=*x*,*y*,*z*)
$\sigma_{ij}^{\infty},\tau_{ij}^{\infty}$	: Stress at the infinity (*i*,*j*=*x*,*y*,*z*)
a, *b*	: Major and minor axes of cracks respectively
K_n	: Stress intensity factor (n=I,II,III)
F_n	: Normalized stress intensity factor (n=I,II,III)
NT	: Number of triangles
s	: Distance from the crack surface along *x*-axis
e	: Distance between cracks surface

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