Journal of Engineering Science 08(1), 2017, 103-108

JES
an international Journal

SOME POSTULATES ON SPECTRUM OF EIGEN VALUES REGARDING MODIFIED POWER METHOD

A. R. M. Jalal Uddin Jamali* and Md. Sah Alam

Department of Mathematics, Khulna University of Engineering and Technology, Bangladesh.

Received: 16 May 2017 Accepted: 29 June 2017

ABSTRACT

Eigen value problems arise naturally from a wide variety of scientific and engineering applications. There exist some smart methods in literature where only the Eigen values were found but not the corresponding Eigen vectors.Power method is very simple but a powerful tool for finding largest Eigen value and corresponding Eigen vector (Eigen-pair). On the other hand Inverse Power method is applied to find out smallest Eigen-pair and/or desire Eigen-pairs. But it is known that inverse power method is computationally very costly. Again by using shifting property in Power method,we can find further Eigen-pairs. Exploiting the shifting property in Power Method, we have proposed four postulatesconcerning Eigen spectrum. Moreover incorporating thesepostulates upon Power method, we have proposed a modified algorithm of power method. Typical examples are presented to demonstrate the validity of the postulates as well as the proposed algorithm. By exploiting the postulates, themodified algorithm of power method is able to find out both (largest and smallest) Eigen-pairs successfully in some relevant cases.

KeyWords:*Eigen value, Eigen vector, Power method, Inverse Power method, Modified Algorithm of Power Method.*

1. INTRODUCTION

The theory and computation of Eigenvalue and Eigen vectors are the most successful and widely used tools of applied mathematics and scientific computing. Matrix Eigen value problems arise naturally from a wide variety of scientific and engineering applications such common applications including structural dynamics, quantum chemistry, quantum mechanics, electrical networks, control theory and design, material science, the vibrations of membranes, in the separation of variables for the problems of heat conduction or acoustics or in the hydrodynamic stability analysis (Chu and Golub2002), (Mehrmann and Schroder, 2011). Acoustics, earthquake engineering, Markov chains, pattern recognition, graph theory, stability analysis, the dynamics of elastic bodies, the physics of rotating bodies, small oscillations of vibrating systems, system identification, seismic tomography, principal component analysis, exploration and remote sensing, antenna array processing, geophysics, molecular spectroscopy, particle physics, structure analysis, circuit theory, Hopfield neural networks, mechanical system simulation and many other areas[(Michiels and Niculescu,2008; Sachdev, 1999; Singh and Ram, 2002 and Unger. G, 2013).

Eigen values are a special set of scalars associated with a linear system of equations (i.e., a matrix equation) that are sometimes also known as characteristic roots, characteristic values, proper values, or latent roots (Hoffman and Kunze, 1971). Each Eigen value is paired with a correspondingEigenvector. Mathematically, theEigen value problem forsquare matrices **A**is defined as follows

$\mathbf{Ax} = \lambda\mathbf{x}$ (1)

where λ is called Eigen value and $\mathbf{x}$ ($\neq 0$) is called corresponding Eigen vector. The schematic view of Eigen value problem is illustrated in Fig.1; where $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.Then we have ($\lambda$, $\mathbf{x}$) = {(3,[1,1]/), (-1,[1,-1]/)}.

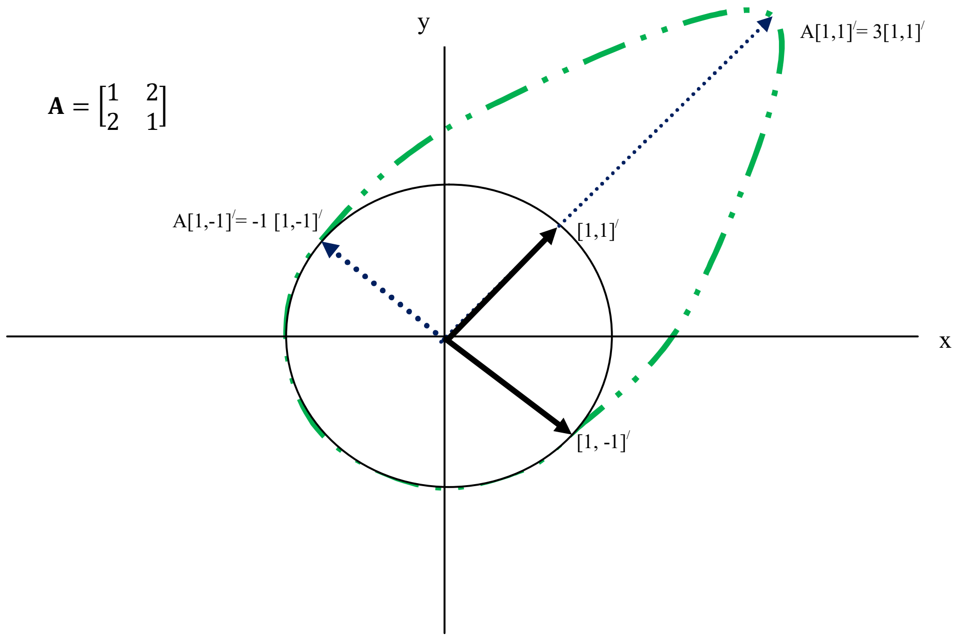

Figure 1: Geometry of Eigen value Problem

Some efficient methods are available in the literature but they are able to find out only Eigen values.For finding corresponding Eigen vectors, much more efforts are necessary.The best known direct method, which is able to find all Eigen values, is the QR-Algorithm, based on the QR decomposition of a matrix which is also implemented in the Matlab function **eig( ).** The QR Iteration algorithm for computing the Eigen values of a general matrix came from an elegantly simple idea that was proposed by Heinz Rutishauser in 1958 and refined by Francis in 1961-1962(Jian, 2004 and Liao *et. al.*2010). But this method is failed to find out corresponding Eigen vectors.

Power method is applicable in various field of application where only largest Eigen-pairs (Eigen value and corresponding Eigen vector) are important. On the other hand Inverse Power method is used to find out smallest Eigen-pair. Moreoverusing shifting property, Inverse Power method is also applicable to find out desire Eigen-pair. It is important to note that various areas of science and engineering seek multiple Eigen-pairs (largest as well as smallest Eigen-pairs) for the reasons other than algorithmic gains.

Besides Power method and Inverse Power method, there are several methods available in the literature for finding both Eigen values and Eigen vectors. Chang and Wong (2008) proposed several refinements of the Power method that enable the computation of multiple extremely Eigen-pairs of very large matrices by using Monte Carlo simulation method. Panju,2011 examined some numerical iterative methods for computing the Eigen values and Eigen vectors of real matrices. He examined five methods – from the simple Power iteration method to the more complicated QR iteration method. The derivations, procedure, and advantages of each method are briefly discussed in that paper. Jamali and Sah Alam,2015proposed modified iterative method based on Power method and Inverse Power method for finding largest and smallest Eigen pairs.

2. PROPOSED POSTULATES

Before state the postulates, it is worthwhile tointroduce some properties of Power method which are related with the proposed postulates.

Property (a)(Shifting Property): if (λ, **x**) be any Eigen-pair of equation (1) and α be any scalar quantity, then (λ- α)be an Eigen value of $(\mathbf{A} - \alpha\mathbf{I})$. So we have

$(\mathbf{A} - \alpha\mathbf{I})\mathbf{x} = (\lambda - \alpha)\mathbf{x}$(2)

Here(λ- α, **x**) be the Eigen-pair of the shifting Eigen problem (2).
Property (b): The Eigen value obtained by Power method is largest in magnitude. This Eigen value is denoted here as *first*Eigen value.
Property (c): Using shifting property, shifting with first(largest) Eigen value, the Power method is able to find out another Eigenvalue and corresponding Eigen vector. This Eigen value is termed as *second*Eigen value.
Now the postulates,which are associated with the Power method as well as modified (using shifting property) Power method, have been given below.

Postulate 1. If *first* (largest) Eigen value is positive and *second*Eigen value is also positive (produced by shifting largest one) then *first*Eigen value be the largest both in magnitude as well as in actual value. On the other hand the *second*Eigen value is the smallest both in magnitude as well as in actual value. In consequence, all Eigen values are positive in sign.

Postulate 2. If *first* (largest) Eigen value is positive and *second*Eigen value is negative (produced by shifting largest one) then the *first*Eigen value be the largest both in magnitude as well as in actual value. On the other hand the *second*Eigen value (obtained by the algorithm) is the smallest in actual value but it is the largest in magnitude among all negative Eigen values (if any). In consequence, some Eigen values along with largest Eigen value are positive and some Eigen values are negative in sign (if exist).

Postulate 3. If *first* (largest) Eigen value is negative and *second*Eigen value is also negative (produced by shifting largest one) then *first*Eigen value be the largest in magnitude but smallest in actual value. On the other hand the *second*Eigen value is the smallest in magnitude but largest in actual value. In consequence, all Eigen values are negative in sign.

Postulate 4. If *first* (largest in magnitude) Eigen value is negative and *second*Eigen value is positive (produced by shifting largest one) then *first*Eigen value be the largest in magnitude but smallest in actual value. On the other hand the *second*Eigen value is the largest in actual value and it is also largest among all positive Eigen values (if any). In consequence, some Eigen values along with largest Eigen values are negative in sign and some Eigen values are positive in sign (if exist).

3. POWER METHOD AND MODIFIED POWER METHOD

By exploiting the postulates upon the Power method, we will develop an algorithm to find out Eigen-pairs. As the proposed algorithm is developed on the basis of Power method, so it is worthwhile to present the algorithm of Power method. Table 1 represents the algorithm of Power method.

```
Power Method ( ) {
    Step (1): read A
              set  y = x0
              set  ξ= ξ0
              set  Imax
for k = 1,2, ……, Imax do {
   Step (2):      v = y / ||y||2
    Step (3):      y = Av
    Step (4):      θ = v * y
    Step (5):      if ||y – θ v||2 ≤ξ |θ|,
           set (λ, x )= (θ, v)
else continue
}end for
    Step (6):    accept (λ, x )= (θ, v)
}
```

Table 1: Algorithm of Power method

Now we will develop the modified power method by incorporating the shifting property. Actually in first iteration, the proposed algorithm runs the power method to find out the *first*largest Eigen value and corresponding Eigen vector. In second iteration, the algorithm uses shifting property where shifting parameter is the first Eigen value. Then, the algorithm again implement Power method on shifted Matrix and able to find out the second Eigen value which is off course the largest Eigen value of the shifted matrix and corresponding Eigen vector. Now transfer the second Eigen value with shifting parameter and this is beinganother (second) Eigen value of the original matrix but Eigen value of vector remains unchanged. Now algorithm compares the signs of the two Eigen values and thencomment about the two Eigen values as well as the sign of the spectrum of Eigen values according to the postulates. A pseudo code of the proposed modified algorithm of Power method is given below:

```
Modified Algorithm of Power Method( )
{
    Step (1):   read A
SetB=Aand {λ, x}={λ0, x0}
for r =1, 2   do{
if   r=1{
Step (2):          apply Power method ( )
                        output  {λ1, x1}
    Step (3) :          find s1, such that  λ1 =s1|λ1|
Step (4):          output {λ1, x1 ,s1}
r = r+1
                   }
else if r =2 {
Step (5): set B=A- λ1I
Step (6):          apply Power method ( )
                        output  {σ2, y2}
    Step  (7) :         λ2 = σ2+ λ1
                        find s2, such that  λ2 = s2|λ2|
Step (8):          output {λ2, x2, s2}
                   }
             }
             end for
    Step (9) :  if (s1 = s2 and > 0)
                   {
output : {(λ1, x1),                  // Largest (in magnitude) Eigen pair
(λ2, x2), // Smallest (in magnitude) Eigen pair
(all λi≥ 0)}           // All Eigen values are positive
    Step (10) :         Stop
                   }
else if (s1= s2 and <0)
                   {
                  output : {(λ1, x1),// Largest (in magnitude) Eigen pair
(λ2, x2),               // Smallest (in magnitude) Eigen pair
(all λi≤ 0)}// All Eigen values are negative
    Step (11) :         Stop
                 }
else if (s1≠ s2 and s1> 0)
                   {
output : {(λ1, x1),                 // Largest (in magnitude) Eigen pair
(λ2, x2),              // Not smallest (in magnitude) Eigen pair
(sign of all λi )}     // Eigen values are both positive and negative
   Step (12) :          Stop
                 }
                   else if (s1≠ s2 and s1<0)
                   {
output : {(λ1, x1), // Largest (in magnitude) Eigen pair
(λ2, x2),               // Not smallest (in magnitude) Eigen pair
( sign of all λi)}// Eigen values are both positive and negative
   Step (13) :         Stop
}
```

4. NUMERICAL EXPERIMENTS

To examine the validity of the firs postulate, we have considered the following typical example 1.

Example 1: $\mathbf{A} = \begin{bmatrix} 4 & 3 & 1 & 0 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 7 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix}$

At first,we have carried out anexperiment on the proposed algorithm. After performing the experiment, the proposed algorithm have obtained the first Eigen-pair = (**10.1375**, [0.498278,0.709471,0.929782,1]$^{/}$)and the second Eigen-pair:(**0.9125**, [-0.913116,1,-0.180717,0.0132391]$^{/}$). Moreover, as the first Eigen is positive and second Eigen value is also positive so according to the postulate 1 the second Eigen value should be thesmallest Eigen value. Moreover as both first and second Eigen values are positive so all real Eigen values of the problem should be positive.

Now to justify the above argument of the postulate as well as to verify the proposed algorithm, we have solved the problem 1 by using MatLab solver to find out the spectrum of Eigen values. From MatLab,we have the following Eigenvalues: **10.1375,** 6.9741, 4.9759and **0.9125**. It is observed that the experimental results agree with the argument of postulate 1. Moreover proposed algorithm is able to find out the desire results. That is the proposed algorithm is successfully able to find out both largest and smallest (in magnitude) Eigen values and corresponding Eigen vectors.

To investigate the validity of the second postulate as well as proposed algorithm, we have considered the following typical example 2.

Example 2: $$\mathbf{B}=\begin{bmatrix}2 & 2 & 1 & 2\\ 8 & 2 & 1 & 4\\ 9 & 2 & 2 & 5\\ 1 & 2 & 5 & 6\end{bmatrix}$$

Again we have carried out numerical experiment on the proposed algorithm. After performing the experiment, the proposed algorithm have obtained the first Eigen-pair = (**12.7568**, [0.418653, 0.77234, 0.958699, 1]$^{/}$)and second Eigen-pair = $(\mathbf{-2.6332}$, [-0.303505, 0.972362, 1, -0.769269]$^{/}$). It is observed that the sign of first Eigen value is positive and the sign of the second Eigen value is negative. So first Eigen value should be positive largest Eigen value and according to the postulate 2, thesecond one should be the smallest Eigen value according to the sign butit should not be necessarilysmallest in magnitude. Moreover as first one is positive and second one is negative so some Eigen value are positive and some Eigen values (if more) should be negative in sign.

Now to justify the above argument, we have solved the problem 2 by using MatLab solver to find out the spectrum of Eigen values. From MatLab, we have the following Eigen values: **12.7568**, 1.4718, 0.4045, $\mathbf{-2.6332}$. It is observed that the largest Eigen value is 12.7568 which is positive and smallest one is -2.6332 which is not smallest in magnitude. Therefore the experimental results agree with the argument of postulate 2.It is noticed that the smallest (in magnitude) Eigen value is ***0.4045***.

Now to justify the postulate 3 and to test the validationof the proposed algorithm we have considered the typical example 3.

Example 3: $$\mathbf{C}=\begin{bmatrix}-26 & -26 & -17 & -4\\ -27 & -30 & -26 & -14\\ -18 & -26 & -55 & -17\\ -16 & -17 & -18 & -66\end{bmatrix}$$

Again after carried out an experiment on the proposed algorithm, we have first Eigen-pair = (**-102.7695**, [0.498278, 0.709471, 0.929782, 1]$^{/}$ and second Eigen-pair =($\mathbf{-0.8326,}$ [-0.913117, 1, -0.180716, 0.0132394]$^{/}$. It is observed that the sign of first the Eigen value is negative and the sign of the second Eigen value is also negative. So first Eigen value should be negative but largestEigen value in magnitude and according to the postulate 3, thesecond one should be the smallest Eigen value according to both in sign and in magnitude. Moreover as first one is negative and second one is also negative so according to the postulate 3, allreal Eigenvalues shouldbe negative in sign.

Now to test the validity of the postulate as well as algorithm, we have solved the problem 3 by using MatLab solver to find out the spectrum of Eigen values. From MatLab, we have the following Eigenvalues:**–102.7695**, -48.6385, -24.7594, $\mathbf{-0.8326}$. It is observed that the largest(in magnitude)Eigenvalue is -102.7695 which isnegative and the smallest one is -0.8326 which is also negative andis the smallest in magnitude too. Therefore the experimental results agree with the argument of postulate 3. In the same time the proposed algorithm is able to find both the largest as well the smallest (in magnitude) Eigen values and corresponding Eigen vectors successfully.

Now for the justification of the postulate 4 and to test the validity of the algorithm,we have considered the following typical example 4.

Example 4: $$\mathbf{D}=\begin{bmatrix}-6 & 2 & 1 & -2\\ 8 & 2 & 1 & 4\\ 9 & 2 & 2 & 5\\ 1 & 7 & 5 & -6\end{bmatrix}$$

Again we have carried out experiment on the proposed algorithm. After performing experiment, we have first Eigen-pair = ($-$**12.4503,** [0.57957, -0.554055, -0.6303, 1]$^{/}$**)** and second Eigen-pair = (**7.9628**, [0.0746226, 0.770871, 1, 0.749899]$^{/}$). It is observed that the sign of first Eigen value is negative and the sign of the second Eigen value is positive. So first Eigen value should be largest(in magnitude) Eigen value and according to the postulate 4, thesecond one should be the largest Eigen value according to the sign but it should not be necessarily smallest and off course not be the largest in magnitude. Moreover as first one is negative and second one is positive so some Eigen value are positive and some Eigen values (if more) should be negative in sign.

Now to justify the above argument, we have solved the problem 4 by using MatLab solver to find out the spectrum of Eigen values. From MatLab, we have the following Eigenvalues:$-$**12.4503**, -4.1452, 0.6327, **7.9628**. It is observed that the largest (in magnitude) Eigen value is -12.4503 which is negative andthe largest positive Eigen values one is 7.9628 which is not smallest in magnitude. Therefore the experimental results agree with the argument of postulate 4.It is noticed that the smallest (in magnitude) Eigen value is ***0.6327***.

5. CONCLUSION

We have proposed four postulates as well as a modified algorithm based on Power iterative method and shifting property. All four postulates are verified by the corresponding typical instances. To investigate the effectiveness of the proposed algorithm, numerical experiments have been carried out. It is observed that proposed algorithm is able to find out the largest Eigen-pair and also smallest Eigen-pairs efficiently according to the relevant postulates 1 and 3. It is observed that in the case of all positive Eigen values as well as in the case of all negative Eigen values the modified Power method successfully is able to find out both largest (in magnitude) as well as smallest (in magnitude) Eigen-pairs. Moreover after finding second Eigen value by the modified Power method, it is easy to classify the nature (regarding sign) of the Eigen values. But it worthwhile to mention here that as the proposed algorithm is developed based on Power method so inherently it preserves some shortcoming of Power methodsuch as in the case of complex Eigen values.

REFERENCE

Chang Q., Wong Y. and Lin C., Journal of Computational Physics, (2008), Vol.19, pp. 8489-8768.

Chu M. and Golub G., Structured Inverse Eigen value Problems, Acta Numer, (2002),Vol. 11, pp. 1-71.

Hoffman K. and R. Kunze, Characteristic Values, Englewood Cliffs, NJ: Prentice-Hall, (1971), pp. 182.

Jamali, A. R. M. Jalal Uddin and Alam Md. Sah, Modified Iterative Algorithms for Eigen-pairs, Proceedings of the 1st International Conference on Mathematics and Its Applications (ICMA - 2015), Khulna University (KU), Khulna, Bangladesh,(2015), pp. 87-91.

Jian B. Z., Numerical Methods for Inverse Eigen value Problems, Ph.D. Thesis, in Mathematics, The Chinese University of Hong Kong, (2004).

Liao B.-S., Bai Z., Lee L.-Q. and Ko K., Nonlinear Rayleigh-Ritz iterative method for solving large scale nonlinear Eigen value problems, Taiwanese J. Math, (2010), Vol. 14(3A) pp. 869–883.

Mehrmann V. and Schroder C., Nonlinear Eigen value and frequency response problems in industrial practice. J. Math. Ind, (2011), Vol. 1(7) pp. 1–18.

Michiels W. and Niculescu S. I., Stability and Stabilization of Time-Delay Systems, Volume 12 of Advances in Design and Control Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. An Eigen value based approach, (2007).

Panju, M., Iterative Methods for Computing Eigen values and Eigenvectors, University ofWaterloo, arXiv:1105.1185, (2011), http:// mathreview.uwaterloo.ca/archive/voli/1/panju.pdf.

Sachdev S., Quantum Phase Transitions, Cambridge University Press, Cambridge, (1999).

Singh K.V. and Ram Y. M., Transcendental Eigen value problem and its applications. AIAA Journal, (2002), Vol. 40(7) pp. 1402–1407.

Unger. G., Convergence orders of iterative methods for nonlinear Eigen value problems. In Advanced Finite Element Methods and Applications, 66 of Lect. Notes Appl. Comput. Mech., Springer, Heidelberg, (2013), pp. 217–237.