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COMPACTNESS IN FUZZY SUPRA TOPOLOGICAL SPACES

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ABSTRACT

The aim of this paper is to introduce and study compactness in fuzzy supra topological spaces. Many properties and characterizations of this with concept have also been investigated.

Keywords: *Fuzzy supra Compact, Fuzzy supra topological space, Lower semi continuous functions, α-supra compact.*

1. INTRODUCTION

In 1965 American mathematician Lotfi A. Zadeh introduced the notion of fuzzy set which has useful applications in various fields. The Fuzzy Supra topological space is a generalization and fuzzification of general Supra topological space. In this paper, we introduce compactness in fuzzy supra topological space and also establish a number of characterizations in this regard.

Let X be a non-empty set, and let $I = [0, 1]$. Let I^X denote the set of all mappings $\lambda : X \rightarrow I$. A member of I^X is called a fuzzy subset of X. The union and intersection of fuzzy sets are denoted by the symbol $\vee$ and $\wedge$ respectively and defined by

$\vee \lambda_i = \max \{ \lambda_i(x) | i \in J \text{ and } x \in X\}$

$\wedge \lambda_i = \min \{ \lambda_i(x) | i \in J \text{ and } x \in X\}$ where J is an index set.

Definition: 1.1. Let X be a non-empty set, and $I = [0, 1]$. A subfamily t^* of I^X is said to be fuzzy supra topology on X.(Monsef and Ramadan 1987) if

(1) $0, 1 \in t^*$

(2) $\alpha_i \in t^*$ for all $i \in J$ then $\vee \alpha_i \in t^*$.

(X, t^*) is called a fuzzy supra topological space, in short **fsts**. The elements of t^* are called fuzzy supra open set and their complements are called fuzzy supra closed set. The lower semi continuous functions on $X \times R$ is denoted by symbol $L(\mu^{\in})$and defined by $L(\mu^{\in})=\{(x, r): \mu^{\in}(x)>r\}$ where $\beta \subset \omega(t^*)$and $\text{Sup}_{\mu\in\beta}\, \mu \geq \alpha$, $r \in [0, \alpha]=I_\alpha$, $\alpha > \epsilon$.

Definition: 1.2. Let (X, t_1^*), (Y, t_2^*) be two fuzzy supra topological spaces. A mapping, $f: (X, t_1^*) \rightarrow (Y, t_2^*)$ is called fuzzy supra continuous if the inverse image of each fuzzy supra open set in (Y, t_2^*) is t_1^* fuzzy supra open in X.

Definition: 1.3. Let X be a non-empty set and T^* be a supra topology on X, and let $t^* = \omega(T^*)$ be the set of all lower semi continuous functions from (X, T^*) to I with usual topology(Lowen 1977). Thus

$t^* = \omega(T^*) = \{u \in I^X : u^{-1}(r, 1) \in T^*\}$, where $r \in [0, 1)=I_1$.

2. COMPACTNESS PROPERTY OF FSTS

Definition: 2.1. Let (X, t^*) be a **fsts**. A family F of fuzzy supra open sets is a cover of a fuzzy set μ if and only if $\mu \subset \{\vee \mu_i: \mu_i \in F\}$. It is called a cover of X, if $\bigvee_{i=1}^{n} \mu_i = 1$.

Definition: 2. 2. A fuzzy supra topological space (X, t^*) is fuzzy supra compact, if every supra open cover of X by the members of t^* contains a finite sub cover, that is if $\mu_i \in t^*$ for all $i \in J$, (J is an index set) then there are finitely many indices $i_1, i_2, i_3, i_4, i_5, i_6, \ldots, i_n \in J$ such that $\bigvee_{J=1}^{n} \mu_{ij} = 1$.

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Definition: 2. 3. Let (X, t^*) be a **fsts**. and $\alpha \in I$. A collection $T^* \subset I^X$ is said to be a fuzzy supra α-shading if and only if for every point $x \in X$ there exist $\lambda \in T^*$ such that $\lambda(x) > \alpha$.

Definition: 2. 4. Let (X, t^*) be a **fsts**. Let $\alpha \in I$ then (X, t^*) is said to be α-supra compact if every fuzzy supra open α-shading of the space has a finite α-sub shading **(**Lowen, 1977**)**.

Theorem: 2.5. Let (X, t^*) be a fuzzy supra topological space. Then the following conditions are equivalent.

(1) $\{\mu_i\}$, $i \in J$ is a cover of X.

(2) $\bigvee_{i\in J} \mu_i = 1$ where $i \in J$ $\forall$ $x \in X$.

(3) $\bigwedge_{i\in J} \mu_i = 0$ where $i \in J$ $\forall$ $x \in X$.

Proof: (1) $\Rightarrow$(2). It is clear from the definition of a Cover, since $\{\mu_i\}$, $i \in J$ is a cover of X means that $\vee_{i\in J} \mu_i = 1$, where $i \in J$, $\forall$ $x \in X$.

(2) $\Rightarrow$(3) Since $\wedge_{i\in J} \mu_i = \inf\{\mu_i\}$ where $i \in J$, $\forall$ $x \in X$.

$= 1 - \sup\{\mu_i\}$, where $i \in J$, $\forall$ $x \in X$.

$= 1 - 1 = 0$.

(3)$\Rightarrow$(1) From (3) as above it can be shown that $\vee_{i\in J} \mu_i = 1$. Which implies that $\{\mu_i\}$ is a cover of X.

Theorem: 2.6. Let (X, T^*), (Y, S^*) are two fuzzy supra topological spaces, with (X, T^*) fuzzy supra compact. Let $f: X \to Y$ be a fuzzy supra continuous surjection. Then (Y, S^*) is fuzzy supra compact.

Proof: Let $u_i \in S^*$ for each $i \in J$ with $\vee_{i\in J} u_i = 1$.Since f is fuzzy supra continuous, so

$f^{-1}(u_i) \in T^*$ .As (X, T^*) is supra compact, we have for each $x \in X$, $\vee_{i\in J} f^{-1}(u_i)(x) = 1$.

So we see that $\{f^{-1}(u_i)\}$, $i \in J$ is a cover of X. Hence $\exists$ finitely many indices $i_1, i_2, i_3, i_4, i_5, i_6, \ldots, i_n \in I$ such that $\vee_{i\in J} f^{-1}(u_{ij}) = 1$. Let u be a fuzzy set in Y. Since f is a surjection we observe that for any $y \in Y$

$$f(f^{-1}(u))(y) = \text{Sup}\{f^{-1}(u)(z): z \in f^{-1}(y)\}$$

$$= \text{Sup}\{u(f(z)): f(z) = y\} = u(y)$$

so that $f(f^{-1}(u)) = u$. This is true for any fuzzy set in Y. Hence

$$1 = f(1) = f(\vee_{i\in J} f^{-1}(u_{ij})) = \vee_{i\in J} f(f^{-1}(u_{ij})) = \vee_{i\in J} u_{ij}.$$

Therefore (Y, S^*) is fuzzy supra compact

Theorem: 2.7. Let (X, T^*), (Y, S^*) are two fuzzy supra topological spaces, and let $f: X \to Y$ is a fuzzy supra continuous surjection. Let A is a fuzzy supra compact set in (X, T^*). Then $f(A)$ is also fuzzy supra compact in (Y, S^*).

Proof: Let $B = \{Gi : i \in J\}$, where $\{Gi\}$ be a fuzzy supra open cover of $f(A)$. Then by definition of fuzzy supra continuity $A = \{f^{-1}(Gi): i \in J\}$ is the fuzzy supra open cover of A. Since A is fuzzy supra compact, then there exists a finite sub cover of A, that is G_{ik}, k=1, 2, 3, ……… n, such that $A \subseteq \vee_{i=1}^{n} f^{-1}(G_{ik})$.

Hence $f(A) \subseteq f(\vee_{i=1}^{n} f^{-1}(G_{ik})) = \vee_{i=1}^{n} f(f^{-1}(G_{ik})) \subseteq \vee_{k=1}^{n} G_{ik}$.

Therefore $f(A)$ is fuzzy supra compact.

Theorem: 2.8. Let (X, T^*), (Y, S^*) are two fuzzy supra topological spaces. Then the product $(X\times Y, \delta^*)$ is fuzzy supra compact if and only if (X, T^*) and (Y, S^*) are fuzzy supra compact.

Proof: First suppose that $(X\times Y, \delta^*)$ where $\delta^* = \{G_i\times H_i: G_i\in T^* \text{ and } H_i\in S^*\}$ is fuzzy supra compact, then we can define a fuzzy continuous surjection mapping κ_1 and π_1 from $(X\times Y, \delta^*)$ to (X, T^*) and (Y, S^*) respectively. Now by the theorem 2.6; (X, T^*) and (Y, S^*) are fuzzy supra compact.

Conversely, let (X, T^*) and (Y, S^*) are fuzzy supra compact. Since $\delta^* =\{G_i\times H_i : G_i\in T^* \text{ and } H_i\in S^* \text{ for } i\in J\}$ where G_i and H_i are fuzzy supra open set. We claim that $\{G_i: i\in J\}$ is a cover of X, and $\{H_i: i\in J\}$is a cover of Y. That is if $\bigvee_{i\in J} G_i(x)=1$ for all $x\in X$, and if $\bigvee_{i\in J} H_i(y)= 1$ for all $y\in Y$, then $\bigvee_{i\in J} \{(G_i\times H_i)(x, y)\}=\text{Sup}\{\min\{G_i(x), H_i(y)\}\}$. Hence we have finite subset J' of J for which $\bigvee_{i\in J} G_i(x)=1$ or $\bigvee_{i\in J} H_i(y)= 1$. Hence we have $\delta^* =\{G_i\times H_i : G_i\in T^* \text{ and } H_i\in S^* \text{ for } i\in J'\}$is a finite sub cover of $(X\times Y, \delta^*)$. Hence $(X\times Y, \delta^*)$is fuzzy supra compact.

Corollary: 2.9. If $(X_i, \delta_i)_{i\in J}$ is a family of fuzzy supra compact topological spaces then $(\prod_{i\in J} X_i, \prod_{i\in J}\delta_i)$ is also fuzzy supra compact.

Theorem: 2.10 The fuzzy supra topological space $(X, \omega(t^*))$ is fuzzy supra compact if and only if (X, t^*) is supra compact.

Proof: Firstly suppose that (X, t^*) is supra compact, let $\beta\subset\omega(t^*)$ be such that $\text{Sup}_{\mu\in\beta}\,\mu\geq\alpha$. Let the lower semi continuous functions $L(\mu^{\epsilon})=\{(x, r): \mu^{\epsilon}(x)>r\}$ is an supra open set of $X\times R$, $r\in[0, \alpha]=I_\alpha$, $\alpha>\epsilon$ .Now $\text{Sup}_{\mu\in\beta} L(\mu^{\epsilon})\supset X\times I_\alpha$, we know that $X_\alpha\times I_\alpha$ is supra compact. Hence $\exists$ finite subfamily $\beta_1\subset\beta$, which covers $X\Rightarrow$ $(X, \omega(t^*))$ is fuzzy supra compact.

Conversely, suppose fuzzy supra topological space $(X, \omega(t^*))$ is a fuzzy supra compact. Then from definition of fuzzy supra compactness $\exists\ \beta_1\subset\beta$ and $\mu_i\in\beta_1$ such that $\text{Sup}\,\mu_i= 1$. Hence (X, t^*) is supra compact.

Theorem: 2.11. Let (X, t^*) is a fuzzy supra topological compact space then there exist a fuzzy supra compact topology $\omega(t^*)$ in which every closed fuzzy set is also fuzzy supra compact.

Proof: Let α and $\alpha^c\in\omega(t^*)$, and $\beta\subset\omega(t^*)$ such that $\text{Sup}_{\mu\in\beta}\,\mu\geq\alpha$. Now $\alpha^c\in\omega(t^*)\Rightarrow 1-\alpha\in\omega(t^*)$, hence the collection $T(\alpha) = \{(x, r): \alpha(x) <r\}$ is fuzzy supra open in $X\times I$.

Therefore $T(\alpha)^c$ is fuzzy supra compact. Choosing $\epsilon>0$ and taking $\mu^{\epsilon}=\mu+\epsilon$, we have

$\text{Sup}_{\mu\in\beta} L(\mu^{\epsilon})\geq T(\alpha)^c$ so there exist finite subfamily β_0 of β such that $\text{Sup}_{\mu\in\beta} L(\mu^{\epsilon})\geq T(\alpha)^c$. So in $\omega(t^*)$ for which every closed fuzzy set is also fuzzy supra compact. Hence the proof of the theorem is complete.

Theorem: 2.12. Let $0\leq\alpha\leq 1$ then a ***fsts*** (X, t^*) is fuzzy α- supra compact, iff (X, t^*_α) is α- supra compact.

Proof: Let (X, t^*) is fuzzy α- supra compact, Let $\mu = \{\mu_i : i\in\Lambda\}$ be a supra open cover of (X, t^*_α). To show (X, t^*_α) is α- supra compact, we have to prove that every open cover has a finite sub cover. Since μ is a supra cover of (X, t^*_α) then by definition of t^*_α there exist $x\in X$ and $\mu_{i_0}\in\mu$ be such that $\mu_{i_0}(x)>\alpha$. Again by definition of fuzzy α-supra compact of (X, t^*) each μ_i has a sub cover $\mu_{i_{i=1}}^{i=n}$, Hence (X, t^*_α) is α- supra compact.

Conversely let (X, t^*_α) is α- supra compact then by t^*_α shading it is clear that (X, t^*) is fuzzy α- supra compact.

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