Journal of Engineering Science 02(1&2), 2011, 29-35

JES
an International Journal

FINITELY GENERATED n-IDEALS WHICH FROM m-NORMAL LATTICES

Md. Abul Kalam Azad[1*], Md. Bazlar Rahaman[1] and A. S. A. Noor[2]

[1]*Department of Mathematics, Khulna University of Engineering and Technology, Khulna-9203, Bangladesh*
[2]*Department of ECE, East West University, 43 Mohakhali, Dhaka, Bangladesh*

Received: 29 November 2010 Accepted: 15 July 2011

ABSTRACT

A convex sublattice of a lattice, L containing an element $n \in L$ is called an n-ideal. The set of all finitely generated n-ideals is denoted by $F_n(L)$, which is a lattice. A distributive lattice with 0 is called an m-normal lattice if its every prime ideal contains at most m number of minimal prime ideals. In this paper we include several characterizations of those $F_n(L)$ which form m-normal lattices. We also show that $F_n(L)$ is m-normal iff for any $x_0, x_1, \ldots\ldots\ldots, x_m \in L$, with $m(x_i, n, x_j) = n$ implies $\langle x_0 \rangle_n^* \vee \ldots\ldots\ldots \vee \langle x_m \rangle_n^* = L$.

Keywords: Ideals, m-normal lattice, n-ideals and Semilattice

1. INTRODUCTION

Lee (1930) and Lakser (1971) has determined the lattices of all equational subclasses of the class of all pseudocomplemented distributive lattices. They are given by $B_{-1} \subset B_0 \subset \ldots\ldots \subset B_m \subset \ldots\ldots \subset B_\omega$, where all the inclusions are proper and B_ω is the class of all pseudocomplemented distributive lattices, B_{-1} consists of all one element algebra, B_0 is the variety of Boolean algebras while B_m, for $-1 \le m \le \omega$ consists of all algebras satisfying the equation

$$(x_1 \wedge x_2 \wedge \ldots\ldots \wedge x_m)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge x_2 \wedge \ldots\ldots \wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge \ldots\ldots \wedge x_m)^* = 1$$

where x^* denotes the pseudocomplement of x. Thus B_1 consists of all Stone algebras.

Davey (1974) has obtained several characterizations of (sectionally) B_m and relatively B_m lattices. On the other hand Cornish (1974) have studied the lattices analogues to B_m and relatively B_m lattices known as m-normal and relatively m-normal lattices.

A distributive lattice **L** with 0 is called m-normal, if each prime ideal of **L** contains at most m-minimal prime ideals. For an element $n \in L$, a convex sublattice containing n is called an n-ideal. n-ideal generated by a finite number of elements $\mathbf{a_1}, \ldots\ldots, \mathbf{a_r}$ is called a finitely generated n-ideal and denoted by $\langle a_1, \ldots\ldots, a_r \rangle_n$. Set of all finitely generated n-ideals is a lattice denoted by $F_n(L)$. An n-ideal generated by a single element is called a principal n-ideal. Set of all principal n-ideals is denoted by $P_n(L)$.

In this paper we include several characterizations of those $F_n(L)$ which form m-normal lattices. We will show that $F_n(L)$ is m-normal if and only if for any $x_1, x_2, \ldots\ldots x_m \in L$, with $m(x_i, n, x_j) = n$ implies $< x_0 >_n^* \vee \ldots\ldots \vee < x_m >_n^* = L$, which is also equivalent to the condition that for any m+1 distinct minimal prime n-ideals $P_0, \ldots\ldots, P_m$ of **L**, $P_0 \vee \ldots\ldots \vee P_m = L$.

We start the paper with the following result on n-ideals due to Latif and Noor (1994).

Lemma 1.1: For $n \in L$, $F_n(L) \cong (n]^d \times [n)$. ■

Following result is also essential for the development of the paper, which is due to Ali(2000) [Theorem 1.1.12].

Lemma 1.2: Let **I** and **J** be two n-ideals of a distriutive lattice. Then for any $x \in I \vee J$, $x \vee n = i_1 \vee j_1$ and $x \wedge n = i_2 \wedge j_2$ for some $i_1, i_2 \in I$, $j_1, j_2 \in J$, with $i_1, j_1 \ge n$, and $i_2, j_2 \le n$ ■

* Corresponding author: azadmath.azad8@gmail.com

A Prime n-ideal **P** is said to be a minimal prime n- ideal belonging to n-ideal **I** if,

(i) $I \subseteq P$, and

(ii) There exists no prime n-ideal **Q** such that $Q \neq P$ and $I \subseteq Q \subseteq P$

A prime n-ideal **P** of **L** is called a minimal prime n-ideal if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. Then a minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$.

Following lemma due to Davey (1974) [Lemma 2.2] will be needed for further development of this paper. This is the dual of Cornish (1974) [Lemma 3.6]. This also follows from the corresponding result for commutative semigroup due to Kist (1973). So we prefer to omit the proof.

Lemma 1.3: Let **M** be a prime ideal containing an ideal **J**. Then **M** is a minimal prime ideal belonging to **J** if and only if for all $x \in M$, there exists $x' \notin M$ such that $x \wedge x' \in J$. ■

Now we generalize this result for n-ideals.

For $a, b \in L$, $< a, b > = \{x \in L : x \wedge a \leq b\}$ is known as annihilator of **a** relative to **b**, or simply a relative annihilator. It is very easy to see that in presence of distributivity, $< a, b >$ is an ideal of **L**. Again for $a, b \in L$ we define $< a, b >_d = \{x : x \vee a \geq b\}$, which we call a dual annihilator of **a** relative to **b**, or simply a relative dual annihilator. In presence of distributivity of **L**, $< a, b >_d$ is a dual ideal (filter).

For $a, b \in L$ and a fixed element $n \in L$, we define

$$< a, b >^n = \{x \in L : m(a, n, x) \in < b >_n\} = \{x \in L : b \wedge n \leq m(a, n, x) \leq b \vee n\}.$$

We call $< a, b >^n$ the annihilator of **a** relative to **b** around the element **n** or simply a relative n- annihilator. It is easy to see that for all $a, b \in L, < a, b >^n$ is always a convex subset containing **n**. In presence of distributivity, it can be easily seen that $< a, b >^n$ is an n-ideal. For two n-ideals **A** and **B** of a lattice **L**, $\langle A, B \rangle$ denotes $\{x \in L : m(a, n, x) \in B\}$ for all $a \in A$. In presence of distributivity, clearly $\langle A, B \rangle$ is an n-ideal.

Lemma 1.4: Let **M** be a prime n-ideal containing an n-ideal **J**. Then **M** is a minimal prime n-ideal belonging to **J** if and only if for all $x \in M$ there exists $x' \notin M$ such that $m(x, n, x') \in J$.

Proof: Let **M** be a minimal prime n-ideal belonging to **J** and $x \in M$. Then by Noor and Ali (1998) $<< a >_n, J > \not\subseteq M$. So there exists x' with $m(x, n, x') \in J$. such that $x' \notin M$.

Conversely, suppose $x \in M$, then there exists $x' \notin M$ such that $m(x, n, x') \in J$. This implies $x' \notin M$, but $x' \in << x >_n, J >$, that is $<< x >_n, j > \not\subseteq. M$. Hence by Noor and Ali (2000) **M** is a minimal prime n-ideal belonging to **J**. ■

Davey (1974) [Corollary 2.3] used the following result in proving several equivalent conditions on $\mathbf{B_m}$ lattices. On the other hand, Cornish (1974) has used this result in studying n-normal lattices.

Proposition 1.5: Let $\mathbf{M_0}$, ----------, $\mathbf{M_n}$ be n +1 distinct minimal prime ideals. Then there exist $a_0, \ldots\ldots, a_n \in L$ such that $a_i \wedge a_j \in J \quad (i \neq j)$ and $a_j \notin M_j \; j = 0, \ldots\ldots n$. ■

Now we generalize the above result in terms of n-ideals.

Proposition 1.6: Let $\mathbf{M_0}$, ---------, $\mathbf{M_n}$ be m+1 distinct minimal prime n-ideals. Then there exist $a_0, \ldots\ldots\ldots, a_n \in L$ such that $m(a_i, n, a_j) \in J(i \neq j)$ and $a_j \notin M_j (j = 0, \ldots\ldots\ldots, n)$.

Proof: For n=1. Let $x_0 \in M_1 - M_0$ and $x_1 \in M_0 - M_1$. Then by Lemma 1.3, there exists $x_1' \notin M_0$ such that $m(x_1, n, x_1') \in J$. Hence $a_1 = x_1, a_0 = m(x_0, n, x_1')$ are the required elements.

Observe that $m\ (a_0, n, a_1) = m\ (m\ (x_0, n, x_1'), n, x_1)$

$$= (x_0 \wedge\ x_1 \wedge x_1') \vee (x_0 \wedge n) \vee (x_1 \wedge n) \vee (x_1' \wedge n)$$

$$= (x_0 \wedge m\ (x_1, n,\ x_1')) \vee (x_0 \wedge n) \vee (m\ (x_1, n, x_1') \wedge n)$$

$$= m\ (x_0, n, m\ (x_1, n, x_1'))\ .$$

Now $m\ (x_1, n, x_1') \wedge n \le m\ (x_0, n, m\ (x_1, n, x_1')) \le m\ (x_1, n, x_1') \vee n$ and $m\ (x_1, n, x_1) \in J,$ so by convexity $m\ (a_0, n, a_1) \in J$.

Assume that the result is true for n=m–1, and let $M_0, \ldots\ldots, M_m$ be m+1 distinct minimal prime n-ideals. Let $b_j (j = 0, \ldots\ldots, m-1)$ satisfy $m\ (b_i, n, b_j) \in J\ (i \ne j)$ and $b_j \notin M_j$. Now choose $b_m \in M_m - \bigcup_{j=0}^{m=1} M_j$ and by Lemma 1.4, let $b'_m \notin M_m$ and $m\ (b_m, n, b'_m) \in J$. Clearly ,

$a_j = m\ (b_j, n, b_m)\quad (j\ = 0, \ldots\ldots\ldots, m-1)$ and $a_m\ = b'_m$, establish the result. ■

Let J be an n-ideal of a distributive lattice **L**. A set of elements $x_0, \ldots, x_m \in L$ is said to be pairwise in **J** if $m\ (x_i,\ n,\ x_j) = n$ for all $i \ne j$.

The next result is due to Cornish (1974) [Lemma 2.3] which was suggested by Hindman (1972) [Theorem 1.8] .

Lemma 1.7: Let **J** be an ideal in a lattice **L**. For a given positive integer $n \ge 2$, the following conditions are equivalent.

(i) For any $x_1, \ldots\ldots, x_n \in L$ which are "pairwise in **J**" that is $x_i \wedge x_j \in J$ for any $i \ne j$, there exists k such that $x_k \in J$;

(ii) For any ideals $J_1, \ldots, J_n$ in L such that $J_i \cap J_j \subseteq J$, for any $i \ne j$, there exists k such that $J_k \subseteq J$;

(iii) **J** is the intersection of at most n–1 distinct Prime ideals. ■

Our next result is a generalization of above result. This result will be needed in proving the next theorem which is the main result of this section. In fact, the following lemma is very useful in studying those $P_n\ (L)$ which are m-normal.

Lemma 1.8: Let J be an n-ideal in a lattice L. For a given positive integer $m \ge 2$, the following conditions are equivalent :

(i) For any $x_1, x_2, \ldots, x_m \in L$ with $m\ (x_i, n, x_j) \in J$ (that is, they are pairwise in J) for any $i \ne j$, there exists k such that $x_k \in J$;

(ii) For any n-ideals $J_1, \ldots\ldots, J_m$ in L such that $J_i \cap J_j \subseteq J$ for any $i \ne j$, there exists k such that $J_k \subseteq J$;

(iii) J is the intersection of at most m – 1 distinct prime n-ideals.

Proof: (i) and (ii) are easily seen to be equivalent.

(iii)$\Longrightarrow$(i). Suppose $P_1, P_2, \ldots, P_k$ are k $(1 \le k \le m-1)$ distinct prime –n-ideals such that $J = P_1 \cap \ldots\ldots \cap P_k$. Let $x_1,\ x_2, \ldots\ldots, x_m \in L$ be such that $m\ (x_i, n, x_j) \in J$ for all $i \ne j$. Suppose no element x_i is a member of **J**. Then for each $r(1 \le r \le k)$ there is at most one $i(1 \le i \le m)$ such that $x_i \in P_r$. Since k < m, there is some i such that $x_i \in P_1 \cap P_2 \cap .... \cap P_k$.

(i) $\Longrightarrow$ (iii). Suppose (i) holds for n=2, then it implies that **J** is a prime n-ideal. Then (iii) is trivially true. Thus we may assume that there is a largest integer **t** with $2 \le t < m$ such that the condition (i) does not hold for **J** (consequently condition (i) holds for t+1, t+2, --- m). Then for t<m, we may suppose that there exist elements $a_1, a_2, ..., a_t \in L$ such that $m(a_i, n, a_j) \in J$ for $i \ne j, i = 1, 2, ..., t$ $j = 1, 2, ..., t$ yet $a_1, a_2, ..., a_t \notin J$.

As **L** is a distributive lattice, $<< a_i >_n, J >$ is an n-ideal for any $i \in \{1, 2, ..., t\}$. Each $<< a_i >_n, J >$ is in fact a prime n-ideal. Firstly $<< a_i >_n, J > \ne L$, since $a_i \notin J$. Secondly, suppose that **b** and **c** are in **L** and $m(b, n, c) \in << a_i >_n, J >$. Consider the set of t+1 elements $\{a_1, a_2, ..., a_{i-1}, m(b, n, a_i), m(c, n, a_i), a_{i+1}, ..., a_t\}$. This set is pairwise in **J** and so, either $m(b, n, a_i) \in J$ or $m(c, n, a_i) \in J$ since condition (i) holds for t+1. That is, $b \in << a_i >_n, J >$ or $c \in << a_i >_n, J >$ and so $<< a_i >_n, J >$ is prime.

Clearly, $J \subseteq \bigcap_{1 \le i \le t} << a_i >_n, J >$. If $w \in \bigcap_{1 \le i \le t} << a_i >_n, J >$ then $w, a_1, a_2, ..., a_t$ are pairwise in **J** and so $w \in J$. Hence $J = \bigcap_{1 \le i \le t} << a_i >_n, J >$ is the intersection of t(<m) prime n-ideals . ■

An ideal $J \ne L$ satisfying the equivalent conditions of Lemma 1.7 is called an m-prime ideal. Similarly, an n-ideal $J \ne L$ satisfying the equivalent conditions of Lemma 1.8 is called an m-prime n-ideal.

Now we generalize the Proposition 3.1 of Davey (1974).

Theorem 1.9: Let **J** be an n-ideal of a distributive lattice **L** . Then the following conditions are equivalent:

(i) For m+1 distinct prime n-ideals $P_0, P_1, ..., P_m$ belonging to **J**, $P_0 \vee P_1 \vee .... \vee P_m = L$;

(ii) Every prime n-ideal containing **J** contains at most m distinct minimal prime n-ideals belonging to **J**;

(iii) If $a_0, a_1, ..., a_m \in L$ with $m(a_i, n, a_j) \in J (i \ne j)$ then $\vee_j << a_j >_n, J > = L$.

Proof: (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii). Assume $a_0, a_1, ..., a_m = L$ with $m(a_i, n, a_j) \in J$ and $\vee_j << a_j >_n, J > \ne L$. It follows that $a_j \notin J$, for all j. Then by Latif and Noor (1994) there exists a prime n-ideal **P** such that $\vee_j << a_j >_n, J > \subseteq P$. But by Noor and Ali (1998) we know that **P** is either a prime ideal or a prime filter. Suppose **P** is a prime ideal. For each j, let $F_j = \{x \wedge y : x \ge a_j, x, y \ge n, y \notin P\}$. Let $x_1 \wedge y_1, x_2 \wedge y_2 \in F_j$. Then $(x_1 \wedge y_1) \wedge (x_2 \wedge y_2) = (x_1 \wedge x_2) \wedge (y_1 \wedge y_2)$. Now $x_1 \wedge x_2 \ge a_j$ and $y_1 \wedge y_2 = m(y_1, n, y_2)$. So $t \ge x \wedge y$ implies $t = (t \vee x) \wedge (t \vee y)$. Since $y \notin P$, so $t \vee y \notin P$. Hence $t \in F_j$, and so F_j is a dual ideal. We now show that $F_j \cap J = \phi$, for all j=0, 1, ...m. If not, let $b \in F_j \cap J$, then $b = x \wedge y, x \ge a_j, x \ge n, y \notin P$.

Hence $m(a_j, n, y) = (a_j \wedge n) \vee n \vee (a_j, \wedge y) = (a_j \wedge y) \vee n = (a_j \vee n) \wedge (y \vee n)$.

But $(a_j \vee n) \wedge (y \vee n) \in F_j$ and $n \leq (a_j \wedge y) \vee n \leq b$ implies $m\,(a_j, n, y) \in J$. Therefore, $m\,(a_j\,, n,\, y) \in F_j \cap J$. Again $m\,(a_j, n,\, y) \in J$ with $y \notin P$ implies $<< a_j >_n, J > \not\subseteq P$, which is a contradiction.

Hence $F_j \cap J = 0$ for all j. For each j, let P_j be a minimal prime n-ideals belonging to **J** and $F_j \cap P_j = 0$. Let $y \in P_j$. If $y \notin P_j$, then $y \vee n \notin P$.

Then $m\,(a_j, n,\, y \vee n) = (a_j \vee n) \wedge \; (y \vee n) \in F_j$.

But $m\,(a_j, n,\; y \vee n) \in < y \vee n >_n \subseteq < y >_n \subseteq P_j$, which is a contradiction. So $y \in P$.

Therefore $P_j \subseteq P$, and $a_j \notin P_j$. For if $a_j \in P_j$, then $a_j \vee n \in P_j$. Now, $a_j \vee n \; = (a_j \vee n) \wedge (a_j \vee n \vee y) \in F_j$ for any $y \notin P$.

This implies $P_j \cap F_j \neq \phi$, which is a contradiction. So $a_j \notin P_j$ .But $m\,(a_i, n, a_j) \in J \subseteq P_j (i \neq j)$ which implies $a_i \in P_j (i \neq j)$ as P_j is prime. It follows that $\{P_j\}$ form a set of m+1 distinct minimal prime n-ideals belonging to **J** and contained in **P**. This contradicts (ii). Therefore $\underset{j}{\vee} << a_j >_n, \quad J > = L$.

Similarly, if **P** is filter, then a dual proof of above also shows that $\underset{j}{\vee} << a_j >_n, J > = L$, and hence (iii) holds.

(iii) $\Rightarrow$ (i). Let $P_0, P_1, \ldots\ldots\ldots\ldots, P_m$ be m+1 distinct minimal prime n-ideals belonging to **J**. Then by Proposition 1.6 there exist $a_0, a_1, \ldots\ldots\ldots\ldots\ldots, a_m \in L$ such that $m\,(a_i, n, a_j) \in J\; (i \neq j)$ and $a_j \notin P_j$. This implies $<< a_j >_n, J > \subseteq P_j$ for all j. Then by (iii)

$<< a_1 >_n, J > \vee << a_2 >_n, J > \vee \ldots\ldots \vee << a_m >_n, J > \subseteq P_0 \vee P_1 \vee \ldots\ldots\ldots \vee P_m$, which implies $P_0 \vee P_1 \vee \ldots\ldots\ldots \vee P_m = L$. ■

For a prime n-ideal **P** of a distributive lattice **L**, we write

$n\,(P) = \{\, y \in L \mid m\,(y, n, x) = n$ for some $x \in L - P\}$. Clearly $n\,(P)$ is an n-ideal and $n\,(P) \subseteq P$. Our next result is a nice extension of above result in terms of n-ideals.

Theorem 1.10: Let **L** be a distributive lattice. Then the following Conditions are equivalent :

(i) For any m+1 distinct minimal prime n-ideals $P_0,\ P_1 \ldots\ldots\ldots P_m$,

$$P_0 \vee P_1 \vee \ldots\ldots \vee P_m = L\,;$$

(ii) Every prime n-ideal contains at most m minimal prime n-ideals ;

(iii) For any $a_0 \quad a_1 \ldots\ldots a_m \in L$ with $m\,(a_i n,\, a_j) = n,\; (i \neq j)$

$$i = 0, \ldots.m\,,\; j = 0, \ldots.m\,,\; < a_0 >^* \vee < a_1 >_n{}^* \vee \ldots\ldots \vee < a_m >_n{}^* = L\;;$$

(iv) For each prime n-ideal P, n (P) is an (m +1)- prime n-ideal .

Proof: (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i), easily hold by theorem 1.9 replacing **J** by {n}. To complete the proof we need to show that

(iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv).

(iv) $\Longrightarrow$ (iii). Suppose (iv) holds and x_0 x_1 ...... x_m are m+1 elements of **L** such that $m(x_i, n, x_j) = n$ for $(i \neq j)$. Suppose that $< x_0 >_n^* \vee < x_1 >_n^* \vee .... \vee < x_m >_n^* \neq L$. Then by Stone's separation theorem (Lee 1970) there is a prime n-ideal **P** such that $< x_0 >_n^* \vee < x_1 >_n^* \vee .... \vee < x_m >_n^* \subseteq P$.

Hence $x_0, x_1, ..., x_m \in L - n(P)$. This contradicts (iv) by Lemma 1.8, since $m(x_i, n, x_j) = n \in n(P)$ for all $i \neq j$. Thus (iii) holds.

(ii) $\Longrightarrow$ (iv). This follows immediately from Lemma 1.8. ■

Proposition 1.11: Let **L** be a distributive lattice and $n \in L$. If the equivalent conditions of Theorem 1.10 hold then for any m+1 elements $x_0, x_1, ..., x_m$;

$$(< x_0 >_n \cap < x_1 >_n \cap .... \cap < x_m >_n)^* = \bigvee_{0 \leq i \leq n} (< x_0 >_n \cap .... \cap < x_{i-1} >_n \cap < x_{i+1} >_n \cap .... \cap < x_m >_n)^*$$

Proof : Let $< b_i >_n = < x_0 >_n \cap ...... \cap < x_{i-1} >_n \cap < x_{i+1} >_n \cap ..... \cap < x_m >_n$ for each $0 \leq i \leq m$. Suppose $x \in (< x_0 >_n \cap ...... \cap < x_m >_n)^*$. Then

$$< x >_n \cap < x_0 >_n \cap ...... \cap < x_m >_n = \{n\}.$$ For all $i \neq j$;

$$(< x >_n \cap < b_i >_n) \cap (< x >_n \cap < b_j >_n = \{n\}.$$

So $(< x >_n \cap < b_0 >_n)^* \vee .... \vee (< x >_n \cap < b_m >_n)^* = L$.

Thus $x \in (< x >_n \cap < b_0 >_n)^* \vee .... \vee (< x >_n \cap < b_m >_n)^*$.

Hence by Lemma 1.2, $x \vee n = a_0 \vee .... \vee a_m$ where $a_i \in (< x >_n \cap < b_i >_n)^*$ and $a_i \geq n$, for $i = 0,1,...,m$. Then $x \vee n = (a_0 \wedge (x \vee n)) \vee .... \vee (a_m \wedge (x \vee n))$.

Now $a_i \in (< x >_n \cap < b_i >_n)^*$ implies $< a_i >_n \cap < x >_n \cap < b_i >_n = \{n\}$.

Then by a routine calculation we find that $(a_i \wedge x \wedge b_i) \vee n = n$.

Thus, $< a_i \wedge (x \vee n) >_n \cap < b_i >_n = [n, (a_i \wedge x \wedge b_i) \vee n] = \{n\}$ implies

that $a_i \wedge (x \vee n) \in < b_i >_n^*$ and so $x \vee n \in \langle b_0 \rangle_n^* \vee .... \vee \langle b_m \rangle_n^*$. By a dual proof of above, we can easily show that $x \wedge n \in < b_0 >_n^* \vee ... \vee < b_m >_n^*$. Thus by convexity, $x \in < b_0 >_n^* \vee ..... \vee < b_m >_n^*$. This proves that L. H. S. $\subseteq$ R. H. S.

The reverse inclusion is trivial. ■

Theorem 1.12: For a distributive lattice **L**, the following conditions are equivalent.

(i) $F_n(L)$ is m-normal lattice.

(ii) Every prime n-ideal contains at most m minimal prime n-ideals.

(iii) For any m+1 distinct minimal prime n-ideals P_0, P_1, ……, P_m;

$$P_0 \vee P_1 \vee .... \vee P_m = L.$$

Proof: (i) $\Longrightarrow$ (ii). Let $F_n(L)$ be m-normal. Since $F_n(L) \cong (n]^d \times [n)$, so both $(n]^d$ and $[n)$ are m-normal. Suppose **P** is any prime n-ideal of **L**. Then by Noor and Ali (2000) either $P \supseteq (n]$ or $P \supseteq [n)$. Without loss of generality suppose $P \supseteq (n]$. Then by Noor and Ali (2000), **P** is a prime ideal of **L**. Hence by Lemma 3.4 of

Cornish (1972) $P_1 = P \cap [n)$ is a prime ideal of $[n)$. Since $[n)$ is m-normal, so by Cornish (1974) $\mathbf{P_1}$ contains at most m minimal prime ideals $R_1, R_2, \ldots, R_m$ of $[n)$. Therefore **P** contains m minimal prime ideals $\mathbf{Q_1}, \ldots, \mathbf{Q_m}$ of **L** where $R_i = Q_i \cap [n)$. Since $n \in R_i$, so $n \in Q_i$ and hence $\mathbf{Q_1}, \ldots, \mathbf{Q_m}$ are minimal prime n-ideals of **L**. Thus (ii) holds.

(ii) $\Rightarrow$ (i). Suppose (ii) holds .Let P_1 be a prime ideal in $[n)$. Then by Lemma 3.4of Cornish (1972) $P_1 = P \cap [n)$ for some prime ideal **P** of **L**. Since $n \in P_1 \subseteq P$, so **P** is a prime n-ideal. Therefore by (ii) **P** contains at most m minimal prime ideals $Q_1, \ldots, Q_m$. Thus by Lemma 3.4 of Cornish (1972) P_1 contains at most m minimal prime ideals $R_1, \ldots, R_m$ of $[n)$ such that. $R_1 = Q_i \cap [n)$. Hence by Theorem 1.10 $[n)$ is m-normal. Similarly we can prove that $(n]^d$ is also m-normal. Thus by Lemma 1.1, $F_n(L)$ is m-normal.

(ii) $\Leftrightarrow$ (iii) has already been proved in Theorem 1.10 ■

REFERENCES

Ali, M.A.: A study on n-ideals of a Lattice, Ph.D. Thesis, Rajshahi University, Bangladesh, 2000.

Cornish, W.H.: n- Normal lattices, Proc. Amer. Math. Soc., 1(45), 48-54,1974.

Cornish, W.H.; Normal lattices, J. Austral. Math. Soc., 14, 200-215, 1972.

Davey, B.A.: Some annihilator Conditions on distributive lattices, Algebra Univers. Vol. 4, 3, 316-322, 1974.

Hindman, N.: Minimal n-prime ideal spaces, Math. Ann., 199, 97-114, 1972

Kist, J.E.: Minimal prime ideals in commutative semigroup, Proc. London Math. Soc., 13(3), 13-50, 1973.

Lakser, H.: The structure of pseudocomplemented distributive lattices 1, Sub direct decomposition, Trans. Amer. Math Soc., 156, 335-342, 1971.

Latif, M.A. and Noor, A.S.A.: n-ideals of a lattice, The Rajshahi University Studies (Part B), 22, 173-180, 1994.

Lee, K.B.: Equational classes of distributive pseudocomplemented lattices, Canad. J. Math., 22, 881-891, 1970.

Noor, A.S.A. and Ali, M.A.: Minimal prime n-ideals of a lattice, North Beng. Univ. Rev., India, Vol.9, 1, 1998.

Noor, A.S.A. and Ali, M.A.: Relative annihilators around a neutral element of a Lattice, The Rajshahi University studies (Part B), Vol. 28, 141-146, 2000.